Rational Inattention in the Frequency Domain

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Introduction

This paper solves the canonical dynamic rational inattention tracking problem by formulating it in the frequency domain.

- Main result: rational inattention version of classical Wiener-Kolmogorov filter
  - Admits closed-form expressions in a number of cases
  - Can be solved numerically with a simple iterative algorithm
- Sheds new light on "anticipation effects": rationally inattentive agents care more about allocating attention across frequencies than across time
- Classical approach complements recent state-space approaches in the time domain (Maćkowiak, Matějka, and Wiederholt, 2018; Miao, Wu, and Young, 2020; Afrouzi and Yang, 2020)

Problem

A Gaussian target is driven by structural disturbances,

$$x_t = \sum_{s=-\infty}^{\infty} a_s \varepsilon_{t-s}, \quad \varepsilon_t \sim N(0,1), \quad \sum_{s=-\infty}^{\infty} a_s^2 < \infty$$

Choose an action to solve:

$$\inf_y \; E[(x_t - y_t)'(x_t - y_t)] \quad \text{s.t.}$$

$$\lim_{T\to\infty} \frac{1}{T} I((\varepsilon_{t+1},\dots,\varepsilon_{t+T}),(y_{t+1},\dots,y_{t+T})) \le \kappa \qquad \text{(processing)}$$

$$I((\varepsilon_{t+\tau+1},\varepsilon_{t+\tau+2}\dots), y^t|\varepsilon^{t+\tau}) = 0 \qquad \text{(availability)}$$

Proposition 1. *There exists a solution in which y is Gaussian.*

Frequency domain

Using Gaussianity, write

$$y_t = \sum_{s=-\infty}^{\infty} b_s \varepsilon_{t-s} + v_t, \quad g_s = E v_t v_{t-s}'$$

Turn sequences into functions; e.g. $a(\lambda) = \sum_{s=-\infty}^{\infty} a_s e^{-i\lambda s}$. Then

$$\min_{b,g\ge 0} \frac{1}{2\pi}\int_{-\pi}^{\pi} \text{tr}[(a-b)(a-b)^* + g] d\lambda \quad \text{s.t.}$$

$$\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\frac{|bb^*+g|}{|g|} d\lambda \le \kappa \qquad \text{(processing)}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} b \, d\lambda = 0, \quad s < -\tau \qquad \text{(availability)}$$

Solution

$$\mathcal{L} = \frac{1}{4\pi}\int_{-\pi}^{\pi}\left(\text{tr}[(a-b)(a-b)^* + g - \mu g + 2\psi b^*] + \theta \ln\frac{|bb^*+g|}{|g|}\right) d\lambda$$

Theorem 1. *The solution satisfies*

$$b = \frac{1}{\theta} g(a-\psi) \quad \textit{and} \quad g = \theta U \,\text{diag}\left(\max\left\{1-\frac{\theta}{d_i}, 0\right\}\right) U^*$$

where $(a-\psi)(a-\psi)^* = U\,\text{diag}(d_i)U^*$, and ψ and θ solve

$$\psi = \left[a - \theta U \,\text{diag}\left(\frac{1}{\max\{d_i,\theta\}}\right) U^*(a-\psi)\right]_{-\tau} \qquad (1)$$

$$\theta = \exp\left(-\frac{2\kappa}{n_x} + \frac{1}{2\pi n_x}\int_{-\pi}^{\pi}\sum_{i=1}^{n_x} \ln\max\{d_i,\theta\} d\lambda\right). \qquad (2)$$

Intuition

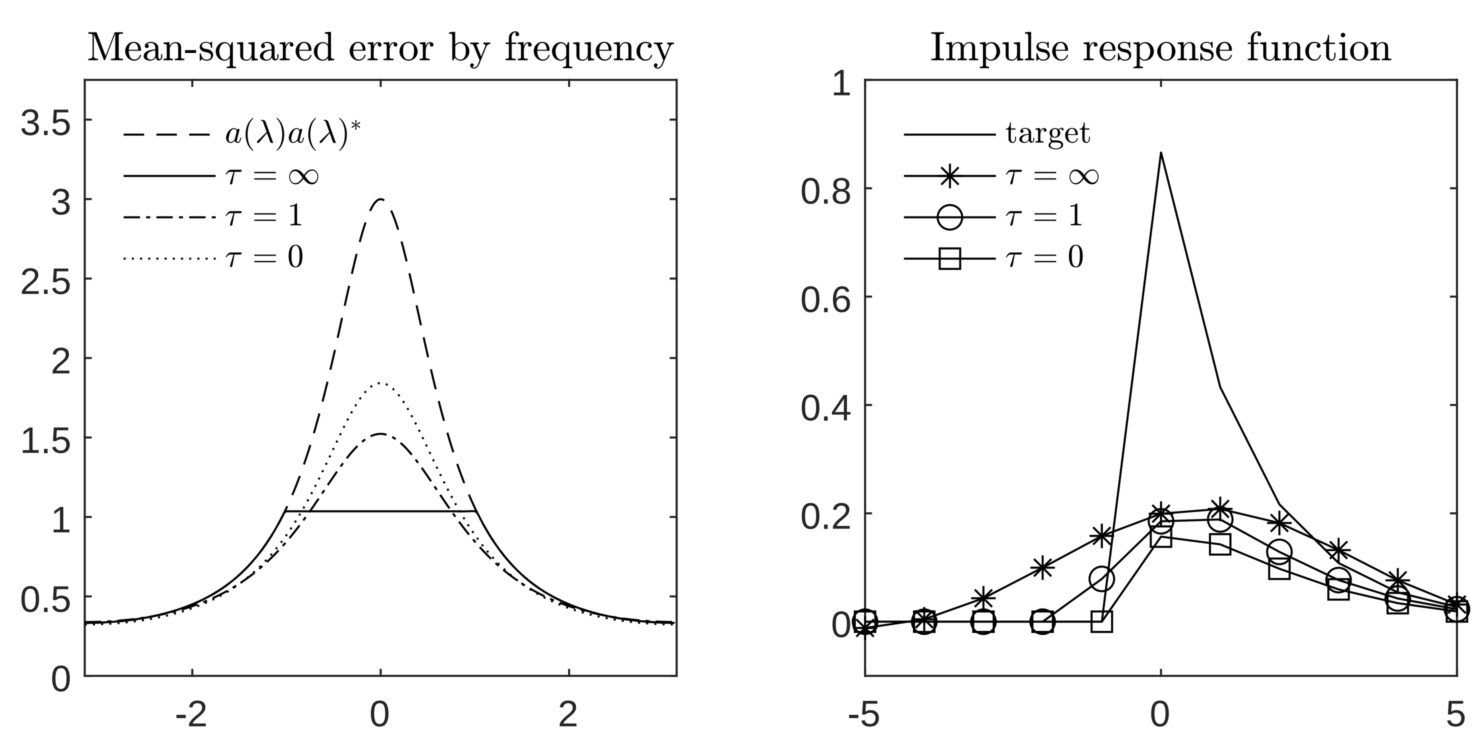
- First best is when $\tau = \infty$; pay attention to most important frequencies, by reverse water-filling on the eigenvalues of the spectral density: $d_i = \text{eig}(aa^*)$
- If $\kappa < \bar{\kappa}$, ignore least important frequencies; implies "anticipation effects"

Proposition 2. *If $\tau = \infty$ and $\kappa < \bar{\kappa}$, then for any s there exists an $s' > s$ such that*

$$E[y_t' \varepsilon_{t+s'}] \neq 0.$$

- Availability constraint prevents this from happening
- When $\kappa \ge \bar{\kappa}$, anticipation effects typically still present; interestingly, only $\tau < \infty$ periods of foresight may be sufficient

Example



Subjective signals

Theorem 1 directly solves for the optimal action $y_t = \sum_{s=-\infty}^{\infty} b_s \varepsilon_{t-s} + v_t$

Proposition 3. *If (i) $\tau < \infty$ or (ii) $\kappa \ge \bar{\kappa}$ and x is regular, then it is always possible to find a subjective signal process s such that*

$$y_t = E[x_t|s^t] \quad \textit{and} \quad s_t = \sum_{k=-\infty}^{\infty} \varphi_k \varepsilon_{t-k} + u_t,$$

where u is uncorrelated over time and independent of ε.

Algorithm

Initialize θ and ψ on a grid, then iterate on (1) and (2). Key steps:

- Use Matlab's `ifft` to evaluate integral in (2)
- Use Matlab's `ifft` and `fft` to evaluate $[\,\cdot\,]_{-\tau}$ in (1)

Advantages	Disadvantages
Allows $\tau = \infty$	Requires stationary target
No state-space form needed	Slower for smaller states
No "curse" in state dimension	

Equilibrium

Monetary misperceptions model of Woodford (2003), with rational inattention:

$$p_{it} = E_{it}[(1-\xi)p_t + \xi q_t]$$

where $p_t = \int p_{it} di$ and $q_t = \sum_{s=-\infty}^{\infty} \delta_s \varepsilon_{t-s}$ is nominal expenditure.

- Target $x_t = (1-\xi)p_t + \xi q_t$ is endogenous when $\xi \neq 1$
- Solve model completely in the frequency domain using a nested loop
- When $\tau > 0$, expansionary stimulus has contractionary effects