The Shadow Price Approach

Bruce McGough, University of Oregon

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The SP approach: motivation

Question: can agents learn to optimize?

- Individually?
- Collectively?

The SP approach: motivation

Question: can agents learn to optimize?

- Individually?
- Collectively?

The SP approach provides a framework for the needed analysis.

The SP approach: motivation

The SP approach: what it is

A paradigm for modeling decision making in dynamic, uncertain environments, based on

- The cognitive consistency principle:

  A model's agents should be neither much smarter than, nor much stupider than the modeler

- The stability principle:

  Optimal/rational behaviors and equilibria are emergent outcomes of stable learning processes

The SP approach: motivation

The SP approach: what it isn't

An attempt to justify REE

- Optimal/rational decision making is a benchmark, not a basis

- Agents are not trying to solve dynamic programming problems

The SP approach: motivation

Talk outline

- Ignore alternative implementations of boundedly rational decision making.

- Introduce shadow-price Bob

- Briefly reference recent applications

- Discuss new preliminary results

Modeling Bob

Let's meet Bob

- Little guy in a big world
- lives in a discrete time, stationary, recursive environment
- each period Bob makes decisions based on
  - the observed state of the world
  - the trade-offs faced, realized and perceived
- Think RBC Bob!

Modeling Bob

Objective primitives: *what the Bob faces*

- endogenous states x (Bob's savings!)
- exogenous states z (prices for Bob! Also, productivity shocks)
- controls u (Bob's consumption and leisure choices!)
- per-period return r (Bob's utility!)
- per-period discount β (Bob's β!)
- endogenous state dynamics g (Bob's budget constraint!)

$$x_{t+1} = g(x_t, z_t, u_t)$$

- exogenous state dynamics f (transitions for prices, etc)

$$z_{t+1} = f(z_t, \varepsilon_{t+1})$$

Modeling Bob

Decision problem: *what Bob does*

- Given states Bob chooses controls
- But to what end, Bob?

Subjective primitives: *what Bob believes*

- λ_t^e is Bob's perception *today* of the marginal value the endogenous state *tomorrow*
- PLMs: $z_{t+1} = Az_t + \hat{\varepsilon}_{t+1}$ and $\lambda_t = H_x x_t + H_z z_t$
- Bob's not the brightest of bulbs
- $\lambda_t^e = H_x E_t^{\text{Bob}} x_{t+1} + H_z E_t^{\text{Bob}} z_{t+1} = H_x g(x_t, z_t, u_t) + H_z A z_t$

Modeling Bob

Behavioral primitives: *what Bob does*

- u and λ^e are determined simultaneously:

$$
\begin{aligned}
0 &= r_u(x_t, z_t, u_t)^\intercal + \beta g_u(x_t, z_{t,t} u)^\intercal \lambda_t^e \\
\lambda_t^e &= H_x g(x_t, z_t, u_t) + H_z A z_t
\end{aligned}
$$

Modeling Bob

Intuition: *how Bob does it*

- contemplates small change in the control: du_{it}
- measures marginal benefit: $r_{u_i} du_{it}$
- measures marginal cost: $-\beta \langle \lambda_t^e, dx_{t+1} \rangle = -\beta g_{u_i}^{\mathsf{T}} \lambda_t^e du_{it}$
- equates them: $r_{u_i} + \beta g_{u_i}^{\mathsf{T}} \lambda_t^e = 0$
- nicely done, Bob

Modeling Bob

Learning

- Regress z_t on lagged z_{t-1} to update A
- Regress λ_t on x_t and z_t to update H, where

$$\begin{aligned} \lambda_t &= r_x(x_t, z_t, u_t)^\intercal + \beta g_x(x_t, z_t, u_t)^\intercal \lambda_t^e \\ &\equiv \lambda(x_t, z_t, H, A) \end{aligned}$$

Modeling Bob

Consistent beliefs

- Fixed beliefs (H,A) yield the recursive system

$$x_{t+1} = g\left(x_t, z_t, u(x_t, z_t, H, A)\right)$$
$$z_{t+1} = f(z_t, \varepsilon_{t+1})$$

  with ergodic distribution ξ

- Project λ on x and z, and z_{t+1} on z_t to get $T(H,A)$

- $T(H^*,A^*) = (H^*,A^*)$ define *consistent beliefs*

Modeling Bob

Consistent beliefs?

Recall the concept of a *restricted perceptions equilibrium* (RPE)

- In RPE, agents use the best forecast rule among those in the restricted class
- An RPE is an equilibrium object: beliefs determine behaviors that must confirm the beliefs
- Misspecification matters
  - omitted variables
  - linear forecasts in a non-linear world

Modeling Bob

Consistent beliefs

- Consistent beliefs is the RPE of decision making
- Consistent beliefs is an equilibrium-like object: beliefs determine behaviors that must confirm the beliefs
- There are many different types of misspecification that could be considered.

"Theorem." Consistent beliefs exist, are first-order optimal, and are stable

Go Bob!

Modeling Bob

What we like

- Satisfies the cognitive consistency principle:
  - Agents use linear forecast rules and RLS
  - Agents solve finite-period problems
- Satisfies the stability principle:
  - Optimal beliefs and behaviors emerge as outcomes of learning processes

▸ Literature review

An RBC economy

Households

- Objective primitives ‣ RBC set-up
  - Return: $u(c_t) - \chi(n_t)$
  - Endogenous state dynamics: $s_t = (1 + r_t)s_{t-1} + w_t \cdot n_t - c_t$
  - Exogenous states: r_t, w_t, v_t
- Subjective primitives
  - PLMs: $\lambda_t = H_0 + H_s s_{t-1} + H_v v_t$

    $v_t - \bar{v} = \rho(v_{t-1} - \bar{v}) + \text{noise}_t$, agent knows the truth
  - Forecasts:

$$\begin{aligned} \lambda_t^e &= E_t^* \lambda_{t+1} = H_0 + H_s s_t + H_v E_t^* v_{t+1} \\ &= H_0 + H_s\left((1 + r_t)s_{t-1} + w_t \cdot n_t - c_t\right) + H_v \rho(v_t - \bar{v}) \\ &= \lambda^e\big(\underset{\substack{\uparrow \\ x_t}}{s_{t-1}}, \underbrace{r_t, w_t, v_t}_{z_t}, \underbrace{c_t, n_t}_{u_t}, H\big) \end{aligned}$$

An RBC economy

Households

- Behavioral primitives: $c_t, n_t, s_t, \lambda_t^e$ solve

$$\begin{aligned} u'(c_t)w_t &= \chi'(n_t) \quad \text{(realized tradeoff)} \\ u'(c_t) &= \beta \lambda_t^e \quad \text{(percieved tradeoff)} \\ \lambda_t^e &= \lambda^e(s_{t-1}, r_t, w_t, v_t, c_t, n_t, H) \\ s_t &= (1+r_t)s_{t-1} + w_t \cdot n_t - c_t \end{aligned}$$

- Optimal beliefs: given processes for r_t and w_t, the household will learn to make first-order optimal decisions

- Of course, in an economic environment r_t and w_t depend on agents' beliefs...

An RBC economy

Temporary equilibrium ▸ Firms

Capital and labor market clearing:

$$
\begin{aligned}
k^d\left(r_t+\delta, w_t, v_t\right) &= \int_{\mathscr{I}} s_{t-1}(i)di \\
n^d\left(r_t+\delta, w_t, v_t\right) &= \int_{\mathscr{I}} n^s\left(s_{t-1}(i), r_t, w_t, v_t, H(i)\right)di.
\end{aligned}
$$

Solving for r_t and w_t provides the temporary equilibrium maps

$$
\begin{aligned}
r_t &= \mathscr{T}\mathscr{E}_r\left(v_t, \{s_{t-1}(i), H(i)\}_{i\in\mathscr{I}}\right) \\
w_t &= \mathscr{T}\mathscr{E}_w\left(v_t, \{s_{t-1}(i), H(i)\}_{i\in\mathscr{I}}\right).
\end{aligned}
$$

▸ Learning dynamics

An RBC economy

Restricted perceptions equilibria

For fixed beliefs H, the economy evolves as

$$\left.\begin{aligned} r_t, w_t &= \mathscr{TE}\left(v_t, k_t, H\right) \\ c_t &= c\left(k_t, r_t, w_t, v_t, H\right) \\ \lambda_t &= (1+r_t)u'(c_t) \\ k_{t+1} &= s\left(k_t, r_t, w_t, v_t, H\right) \\ v_{t+1} - \bar{v} &= \rho\left(v_t - \bar{v}\right) + \sigma\varepsilon_{t+1} \end{aligned}\right\} \leftarrow \text{non-linear!}$$

Let $\hat{H}$ be REE beliefs of the linearized model.

Lemma (EEM) *There exist $\bar{\sigma} > 0$ such that if $\sigma \in (0, \bar{\sigma})$ and H is sufficiently near $\hat{H}$ then*

$$(\lambda_t, k_t, v_t) \xrightarrow{\mathscr{D}} (\lambda, k, v) \sim \xi(H).$$

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Restricted perceptions equilibria

- T-map: $T(H) = \left(E_{\xi(H)} x \cdot x^{\intercal}\right)^{-1} E_{\xi(H)} x \cdot \lambda$, where $x = (1, k, v)^{\intercal}$

- An RPE is a fixed point of the T-map

Theorem (EEM) *If* $\sigma \in (0, \bar{\sigma})$ *then*

- *there exists a unique RPE, denoted* $H^*(\sigma)$
- $H^*(\sigma) \to \hat{H}$ *as* $\sigma \to 0$
- *the RPE is E-stable whenever the linearized REE is E-stable*

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What we like

- Modeler and model's agents on equal footing
  - dynamics are not approximated
  - equilibrium is an emergent outcome learned by modeler and agents in the same way
- REE is a benchmark, but not a basis
- Heterogeneity is accommodated without modification

Figure: RPE stability

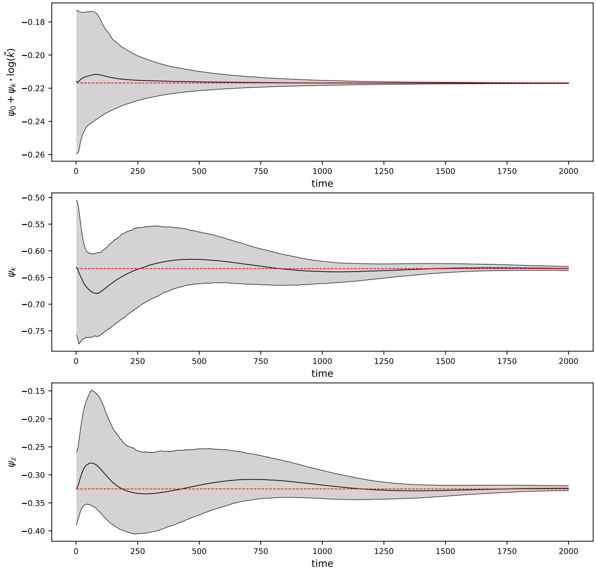


Figure: Consumption schedules

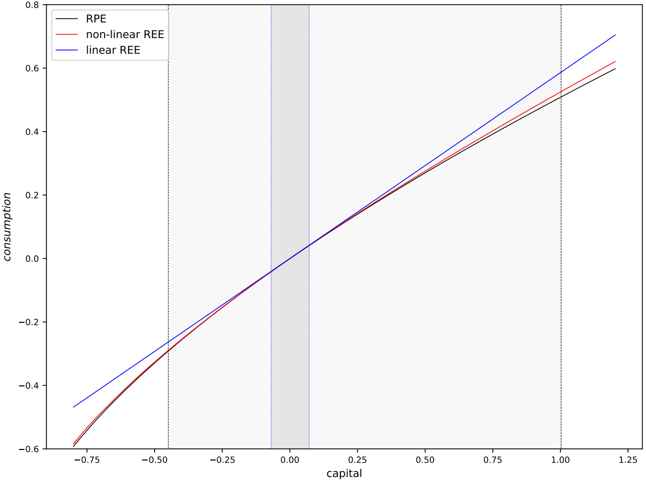
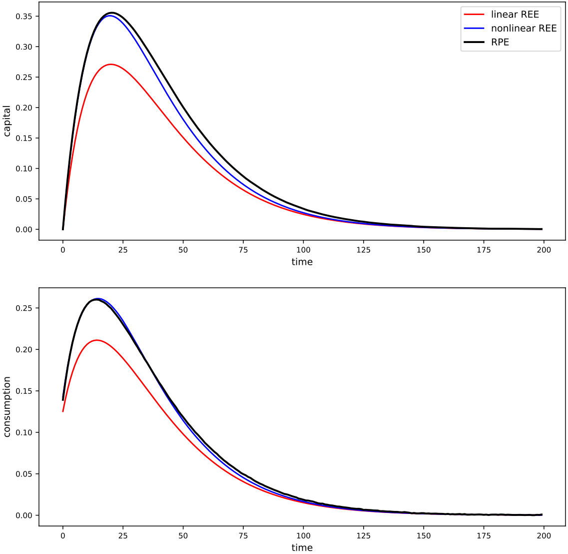


Figure: TFP IRF



Other slides

- Permanent spending shock
- Local rationality

Conclusions

The shadow price approach

- Satisfies cognitive consistency principle
- Satisfies stability principle
- Respects non-linearities
- Accommodates heterogeneity
- May improve model fit

An RBC economy

Players and markets

- Many identical households
  - own capital, time, shares
  - demand goods and leisure
  - supply capital and labor
- Many identical firms
  - own technology
  - demand capital and labor
  - supply goods
- Goods, capital and labor markets are competitive

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An RBC economy

Firms

- Technology: CRTS $y_t = v_t f(k_t, n_t)$, with

$$v_t - \bar{v} = \rho (v_{t-1} - \bar{v}) + \sigma \varepsilon_t$$

- Behavior: max profit given rental-rate $r_t + \delta$ and wage w_t:
  - Demand for capital: $k_t = k^d (r_t + \delta, w_t, v_t)$
  - Demand for labor: $n_t = n^d (r_t + \delta, w_t, v_t)$,

Decision schedules of household $i \in \mathscr{I}$

Demand for goods: $c_t(i) = c(s_{t-1}(i), r_t, w_t, v_t, H(i))$

Demand for savings: $s_t(i) = s(s_{t-1}(i), r_t, w_t, v_t, H(i))$

Supply of labor: $n_t(i) = n^s (s_{t-1}(i), r_t, w_t, v_t, H(i))$

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Dynamics: homogeneous case

Given beliefs H_{t-1} and R_{t-1}, and states k_t and v_{t-1}:

$$\begin{aligned}
v_t &= \bar{v}(1-\rho) + \rho v_{t-1} + \sigma \varepsilon_t \\
r_t, w_t &= \mathscr{TE}\left(v_t, k_t, H_{t-1}\right) \\
\lambda_t &= (1+r_t)u'\left(c\left(k_t, r_t, w_t, v_t, H_{t-1}\right)\right) \\
\xi_t &= (1, k_t, v_t)' \\
R_t &= R_{t-1} + \gamma_t\left(\xi_t \xi_t^{\mathsf{T}} - R_{t-1}\right) \\
H_t &= H_{t-1} + \gamma_t R_t^{-1} \xi_t \left(\lambda_t - H_{t-1}^{\mathsf{T}} x_t\right)
\end{aligned}$$

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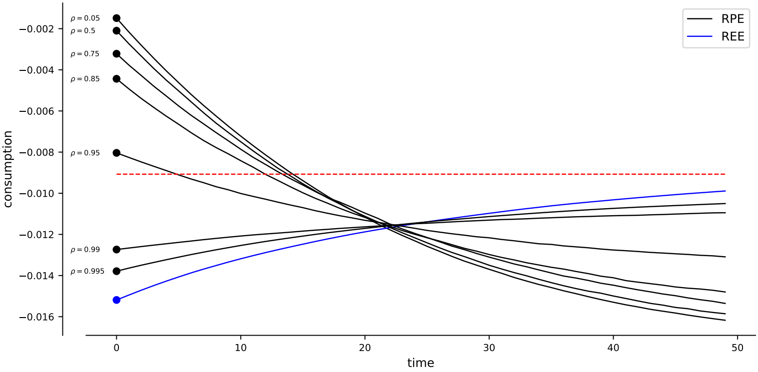
Figure: Permanent spending shock



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Local Rationality

Table: Representative agent model

	Data	RE	$\gamma = 0.001$	$\gamma = 0.01$	$\gamma = 0.035$
$\frac{\text{std}(C)}{\text{std}(Y)}$	0.50	0.32	0.32	0.33	0.34
$\frac{\text{std}(I)}{\text{std}(Y)}$	2.73	3.10	3.09	3.08	

Table: Heterogeneous agent model

	Data	RE	$\gamma = 0.001$	$\gamma = 0.01$	$\gamma = 0.035$
$\frac{\text{std}(C)}{\text{std}(Y)}$	0.50	0.36	0.70	0.63	0.50
$\frac{\text{std}(I)}{\text{std}(Y)}$	2.73	2.91	1.88	2.10	2.50

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Some recent applications

VF-approach

- Branch and McGough, "Heterogeneous beliefs and trading inefficiencies" (JET, 2016)
- Evans, D., Evans, G. and McGough, "Learning when to say no" (JET, forthcoming)

Some recent applications

SP-approach

- Evans, G. and McGough, "Equilibrium selection, observability and backward-stable solutions" (JME, 2018)
- Evans, G. and McGough, "Stable, near-rational sunspot equilibria," (JET, 2020)

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