DOWNSIDE RISK – IMPLICATIONS FOR FINANCIAL MANAGEMENT

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PRAGUE MARCH 2005
RISK AND RETURN

THE TRADE-OFF BETWEEN RISK AND RETURN IS THE CENTRAL PARADIGM OF FINANCE.

HOW MUCH RISK AM I TAKING?
HOW SHOULD I RESPOND TO RISKS THAT VARY OVER TIME?

HOW SHOULD I RESPOND TO RISKS OF VARIOUS MATURITIES?
DOWNSIDE RISK

THE RISK OF A PORTFOLIO IS THAT ITS VALUE WILL DECLINE, NOT THAT IT WILL INCREASE HENCE DOWNSIDE RISK IS NATURAL.

MANY THEORIES AND MODELS ASSUME SYMMETRY: c.f. MARKOWITZ, TOBIN, SHARPE AND VOLATILITY BASED RISK MANAGEMENT SYSTEMS.

DO WE MISS ANYTHING IMPORTANT?
MEASURING DOWNSIDE RISK

Many measures have been proposed. Let $r$ be the one period continuously compounded return with distribution $f(r)$ and mean zero. Let $x$ be a threshold.

- Skewness = $\frac{E(r^3)}{E(r^2)^{3/2}}$
- Probability of loss = $P(r < x)$,
- Expected loss = $E(r \mid r < x)$
- $x$ is the $\alpha$ Value at risk if $P(r < -x) = \alpha$
MULTIVARIATE DOWNSIDE RISK

WHAT IS THE LIKELIHOOD THAT A COLLECTION OF ASSETS WILL ALL DECLINE?

THIS DEPENDS PARTLY ON CORRELATIONS

FOR EXTREME MOVES, OTHER MEASURES ARE IMPORTANT TOO.
Probability that the portfolio loses more than $K$

\[ W_1 P_1 + W_2 P_2 = -K \]
Put Option on asset 1 Pays

Option on asset 2 Pays

Both options Payoff
Symmetric Tail Dependence

$P_{2,T}$

$P_{1,T}$
Lower Tail Dependence

$P_{2,T}$

$P_{1,T}$
Put Option on asset 1
Pays

Option on asset 2
Pays

Both options Payoff

$K_1$

$K_2$

$P_{1,T}$

$P_{2,T}$
CONTAGION

WHERE ARE MY CORRELATIONS WHEN I NEED THEM?

WHEN COUNTRIES DECLINE TOGETHER MORE THAN CAN BE EXPECTED FROM THE NORMAL CORRELATION PATTERN, IT IS CALLED CONTAGION.

CORRELATIONS AND VOLATILITIES APPEAR TO MOVE TOGETHER.
CREDIT DERIVATIVES

- It is well documented that the multivariate normal density underprices joint extreme events such as defaults.
- Industry has adopted a T-copula to price credit baskets and CDO’s.
- Tail dependence is essential in these models.
THE PURPOSE OF MY TALK TODAY

TIME SERIES ANALYSIS OF DOWNSIDE RISK
PURPOSE OF MY TALK TODAY

TO SHOW HOW DOWNSIDE RISK CAN BE MODELED AS A TIME SERIES PROCESS
USING SIMPLY TIME AGGREGATION OF STANDARD TIME SERIES MODELS

CONSEQUENTLY

DOWNSIDE RISK CAN BE PREDICTED
DYNAMIC HEDGING AND DYNAMIC PORTFOLIO STRATEGIES CAN BE IMPLEMENTED.
AN ECONOMETRIC FRAMEWORK

MODEL THE ONE PERIOD RETURN AND CALCULATE THE MULTI-PERIOD CONDITIONAL DISTRIBUTION

RETURN FROM $t$ UNTIL $t + T$ IS:

$$R_t^T = \sum_{j=t+1}^{T+t} r_j$$

THE DISTRIBUTION CONDITIONAL ON TODAY’S INFORMATION IS:

$$R_t^T | F_t \sim f_t^T (R_t^T)$$
ALL MEASURES CAN BE DERIVED FROM THE ONE PERIOD DENSITY

EVALUATE ANY MEASURE BY REPEATEDLY SIMULATING FROM THE ONE PERIOD CONDITIONAL DISTRIBUTION:

METHOD:

- Draw $r_{t+1}$
- Update density and draw observation $t+2$
- Continue until $T$ returns are computed.
- Compute measure of downside risk

$$f_t(r_{t+1})$$
A MODEL

\[ r_t = \sqrt{h_t} \varepsilon_t, \quad \varepsilon_t \sim i.i.d. \]

\[ E_{t-1}(r_t) = 0, \quad h_t = V_{t-1}(r_t) \]

ASYMMETRY FOLLOWS FROM ASYMMETRY IN EPSILON

HOWEVER FOR MULTI-PERIOD RETURNS, THERE IS ANOTHER SOURCE – ASYMMETRIC VOLATILITY.
The ARCH Model

- The ARCH model of Engle(1982) is a family of specifications for the conditional variance.
- The $q^{th}$ order ARCH or ARCH($q$) model is
  \[ h_t = \omega + \sum_{j=1}^{q} \alpha_j r_{t-j}^2 \]
- Notice that it is a simple generalization of both constant variance and rolling variance estimates called "historical volatilities".
The Generalized ARCH model of Bollerslev (1986) is an ARMA version of this model. GARCH(1,1) is a weighted average of three volatility forecasts:

$$h_t = \omega + \alpha r_{t-1}^2 + \beta h_{t-1}$$
Asymmetric Volatility

Often negative shocks have a bigger effect on volatility than positive shocks.

Nelson (1987) introduced the EGARCH model to incorporate this effect.

I will use a Threshold GARCH or TARCH

\[ h_t = \omega + \alpha r_{t-1}^2 + \gamma r_{t-1}^2 I(r_{t-1} < 0) + \beta h_{t-1} \]
NEW ARCH MODELS

- GJR-GARCH
- TARCH
- STARCH
- AARCH
- NARCH
- MARCH
- SWARCH
- SNPARCH
- APARCH
- TAYLOR-SCHWERT

- FIGARCH
- FIEGARCH
- Component
- Asymmetric Component
- SQGARCH
- CESGARCH
- Student t
- GED
- SPARCH
- Autoregressive Conditional Density
- Autoregressive Conditional Skewness
TWO PERIOD RETURNS

- Two period return is the sum of two one period continuously compounded returns
- Look at binomial tree version
- Asymmetric Volatility gives negative skewness
ANALYTICALLY: TARCH
WITH SYMMETRIC INNOVATIONS

\[ E(r_t + r_{t+1})^3 = E\left(r_t^3 + 3r_t^2 r_{t+1} + 3r_tr_{t+1}^2 + r_{t+1}^3\right) \]

\[ = 0 + 0 + 3E\left(r_t h_{t+1}\right) + 0 \]

\[ = 3E\left[r_t\left(\omega + \alpha r_t^2 + \gamma r_t^2 I_{(r_t<0)} + \beta h_t\right)\right] \]

\[ = 3\gamma E\left(r_t^3 I_{(r_t<0)}\right) < 0 \]

and the conditional third moment is

\[ E_{t-1}(r_t + r_{t+1})^3 = 3\gamma E_{t-1}\left(r_t^3 I_{(r_t<0)}\right) < 0 \]
STYLIZED FACTS
Series: RETUSSP
Observations 12455

Mean 0.000318
Median 0.000375
Maximum 0.090994
Minimum -0.204669
Std. Dev. 0.009179
Skewness -0.926286
Kurtosis 28.00273
Jarque-Bera 326200.8
Probability 0.000000
TRIMMING .001 IN EACH TAIL
(8 DAYS)

Series: RETUSSP
Sample 1 12455 IF Y>LOWTRIM
AND Y<HIGHTRIM
Observations 12431

Mean 0.000338
Median 0.000375
Maximum 0.046486
Minimum -0.045594
Std. Dev. 0.008633
Skewness 0.049617
Kurtosis 5.380668
Jarque-Bera 2940.670
Probability 0.000000
SKEWNESS OF MULTIPERIOD RETURNS

[SKEWNESS OF MULTIPERIOD RETURNS diagram with various skewness measures labeled: SKEW_ALL, SKEW_TRIM, SKEW_PRE, SKEW_POST.]
STANDARD ERRORS

- ARE THESE DIFFERENCES SIGNIFICANT?
- THE INFERENCE IS COMPLICATED BY THE OVERLAPPING OBSERVATIONS AND BY THE DEPENDENCE DUE TO ESTIMATING THE MEAN.
- FROM SIMPLE ROBUST TESTS, SIZE CORRECTED BY MONTE CARLO, THESE ARE SIGNIFICANT.
EVIDENCE FROM DERIVATIVES

- The high price of out-of-the-money equity put options is well documented.
- This implies skewness in the risk neutral distribution.
- Much of this is probably due to skewness in the empirical distribution of returns.
- Data matches evidence that the option skew is only post 1987.
MATCHING THE STYLIZED FACTS

- ESTIMATE DAILY MODEL
- SIMULATE 250 CUMULATIVE RETURNS 10,000 TIMES WITH SEVERAL DATA GENERATING PROCESSES
- CALCULATE SKEWNESS AT EACH HORIZON
SKEWS FOR SYMMETRIC AND ASYMMETRIC MODELS

SKEW_EX  
SKEW_EXS  
SKEW_BOOT_EX  
SKEW_BOOT_EXS
IMPLICATIONS

- Multi-period empirical returns are more skewed than one period returns (omitting 1987 crash)
- Asymmetric volatility is needed to explain this.
- Skewness has increased since 1987, particularly for longer horizons.
- Simulated skewness is noisy because higher moments do not exist when the persistence is so close to one. Presumably this is true for the data too.
- Many other asymmetric models could be compared on this basis.
MULTIVARIATE MODELS
DOWNSIDE RISK RESULTS FROM TIME AGGREGATION WITH:

- ASYMMETRIC CORRELATIONS
  - CORRELATIONS RISE PARTICULARLY AFTER TWO ASSETS BOTH DECLINE. (Asymmetric DCC (Cappiello, Engle, Sheppard(2004))

- VOLATILITY SHOCKS ARE CORRELATED
  - PURE VARIANCE COMMON FEATURES(Engle, Marcucci(2005))
  - FACTOR MODELS (Engle Ng and Rothschild(1992))
  - CREDIT RISK MODEL(Engle, Berd, Voronov(2005))
FACTOR ARCH

- RETURNS ARE DRIVEN BY A SMALL NUMBER OF FACTOR SHOCKS, $f_t$.
- FACTORS DRIVE VOLATILITIES AND CORRELATIONS

\[
\begin{align*}
  r_t &= B f_t + u_t \\
  V_{t-1}(r_t) &= B \Omega_t B + D_t \\
  V_{t-1}(f_t) &= \Omega_t, \quad V_{t-1}(u_t) = D_t
\end{align*}
\]
DOWNSIDE RISK IN THE CAPM

The return on a stock can be decomposed into systematic and idiosyncratic returns using the beta of the stock

\[ r_{i,t} = \beta_i r_{m,t} + \varepsilon_{i,t} \]

If the market declines substantially, many stocks will decline. There will be skewness in each stock and downside risk in the portfolio.
SKEWNESS

Under the standard assumptions, the skewness of return \( i \) is related to the return of the market by \( s_i = s_m R^3 \) where \( R^3 \) is the conventional \( R^2 \) raised to the 3/2 power.

Notice that all stocks will then have skewness but that it will be less than for the market.
The probability that two stocks will both underperform some threshold can be calculated conditional on the market return.

\[
P(r_i < k \text{ and } r_j < k) = E\left( P(r_i < k \text{ and } r_j < k \mid r_m) \right)
\]

\[
= E\left( P(\varepsilon_i < k - \beta_i r_m \mid r_m) P(\varepsilon_j < k - \beta_j r_m \mid r_m) \right)
\]

\[
= P(r_i < k) P(r_j < k)
\]

\[
+ \text{Cov}\left( P(\varepsilon_i < k - \beta_i r_m \mid r_m), P(\varepsilon_j < k - \beta_j r_m \mid r_m) \right)
\]

\[
= E\left( \Phi(k - \beta_i r_m) \Phi(k - \beta_j r_m) \right) \text{ under normality}
\]
SUMMARY

ASYMMETRIC VOLATILITY IN THE MARKET FACTOR IMPLIES

- SKEWNESS IN MULTIPERIOD MARKET RETURNS
- SKEWNESS IN MULTIPERIOD EQUITY RETURNS
- LOWER TAIL DEPENDENCE IN EQUITY RETURNS
IMPLICATIONS FOR FINANCIAL MANAGEMENT
IMPLICATIONS FOR RISK MANAGEMENT

MULTI-PERIOD RISKS MAY BE SUBSTANTIALLY DIFFERENT FROM ONE PERIOD RISKS.

THE MULTI-PERIOD RISK CHANGES OVER TIME AND CAN BE FORECAST.

BIG MARKET DECLINES ARE MORE LIKELY WHEN VOLATILITY IS HIGH.
IMPLICATIONS FOR DERIVATIVE HEDGING

As each new period return is observed, the derivative can be repriced and the hedge updated.

Greens can be calculated from simulation pricing to simplify the updating.
IMPLICATIONS FOR PORTFOLIO SELECTION

- MEAN VARIANCE PORTFOLIO OPTIMIZATION WILL MISS THESE ASYMMETRIES.

- HIGH FREQUENCY REBALANCING WILL GIVE \textit{EARLY WARNING} OF DOWNSIDE RISK.
HOW TO DO THIS?

**SUBOPTIMAL METHOD 1**
- MYOPIC ASSET ALLOCATION ON A HIGH FREQUENCY BASIS.
- AS VOLATILITIES RISE, YOU NATURALLY SHIFT OUT OF RISKY ASSETS.

**SUBOPTIMAL METHOD 2**
- MULTI-PERIOD FORECAST OF RISK GIVES AN EX-ANTE OPTIMAL PLAN.
- OVERINVEST WHEN VOLATILITY IS LOW AND UNDERINVEST WHEN IT IS HIGH
OPTIMAL METHOD

DYNAMIC PROGRAMMING:

• WHEN VOLATILITY IS LOW, UNDERINVEST, RECOGNIZING THAT THIS PLAN MAY CHANGE WHEN THE SUBSEQUENT VOLATILITY IS OBSERVED

• SEE COLACITO AND ENGLE (2004)
EXPECTED RETURNS

- EACH OF THESE METHODS REQUIRES EXPECTED RETURNS.
- THE LISTED IMPLICATIONS ARE BASED ON THE ASSUMPTION THAT EXPECTED RETURNS ARE UNCHANGED.
- IS THIS REASONABLE?
BUT IF EVERYBODY DID THIS?

IF ALL AGENTS FOLLOW THIS PATTERN THEN EXPECTED RETURNS WOULD NECESSARILY ADJUST. RETURNS WOULD INSTANTANEOUSLY MOVE ENOUGH TO RESTORE EQUILIBRIUM. CAMPBELL AND HENTSCHEL (1992)

IN A REPRESENTATIVE AGENT WORLD, THERE WOULD NO LONGER BE A MOTIVE FOR ADJUSTING TO CHANGES IN RISK.

CHANGES IN RISK WOULD LEAD TO UNAVOIDABLE CAPITAL GAINS OR LOSSES.

DERIVATIVE REPLICAION STRATEGIES WOULD CONTINUE TO BE USEFUL.
HOWEVER

- THERE IS NO REASON TO BELIEVE DOWNSIDE RISK WOULD DISAPPEAR OR COLLAPSE TO AN INSTANT IN TIME.
- WITH HETEROGENEITY, THERE WOULD STILL BE REASONS TO REBALANCE.
- FROM A MICROSTRUCTURE POINT OF VIEW IT IS DIFFICULT TO IMAGINE HOW THE PRICES COULD INSTANTANEOUSLY ADJUST TO VOLATILITY NEWS.
- EXPECTED RETURNS WOULD BE EXCEEDINGLY DIFFICULT TO ESTIMATE AT THIS HIGH FREQUENCY
- MAYBE WE ARE ALREADY AT THIS POINT SO THAT DOWNSIDE RISK IS FULLY AND INSTANTLY PRICED.
CONCLUSIONS

ASYMMETRIC VOLATILITY AND CORRELATION MODELS ARE POWERFUL TOOLS FOR ANALYZING DOWNSIDE RISK

ONE PERIOD MODELS HAVE BIG IMPLICATIONS ABOUT LONG HORIZON OF RETURNS

THE UPDATING OF VOLATILITY AND RISK MEASURES HAS A NATURAL APPLICATION TO DERIVATIVE HEDGING AND POSSIBLY PORTFOLIO REBALANCING.