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Volha Audzei and Sergey Slobodyan *

Abstract

This paper studies convergence properties, including local and global strong E-stability, of the rational expectations equilibrium (REE) under non-smooth learning dynamics, and the role of monetary policy in agents' expectation formation. In a New Keynesian model, we consider two types of informational constraints that operate jointly - Sparse Rationality under Adaptive Learning. We study the dynamics of the learning algorithm for the positive costs of attention, initialized from the equilibrium with mis-specified beliefs. We find that, for any initial beliefs, the agents' forecasting rule converges either to the Minimum State Variable (MSV) REE, or, for large attention costs, to a rule with anchored inflation expectations. With stricter monetary policy the convergence is faster. A mis-specified forecasting rule that uses a variable not present in the MSV REE does not survive this learning algorithm. We apply the theory of non-smooth differential equations to study the dynamics of our learning algorithm.

JEL Codes: D84, E31, E37, E52.
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1. Introduction

1.1 Sparse Adaptive Learning

Modelling agents expectation formation in self-referential systems has been extensively debated in the literature. The debate has received a different angle when increased computer power and data availability simplified dimensionality reduction and variable selection.[1] With a large stock of predictors available without theoretical justification for their forecasting power, finding a meaningful and intuitive model specification becomes a challenging task. In a dynamic self-referential system, the choice of variables in the forecasting models affects the system itself. A vast Adaptive Learning (AL) literature has studied the conditions under which coefficients, or 'beliefs', of a fixed forecasting model in self-referential systems converge. The convergence has been shown to depend on the stability of an ordinary differential equation approximating the dynamics under learning with particular beliefs updating rule; if the learning uses a Least Squares regression, the concept is called Expectational-, or E-stability. The dynamics and stability of learning in the presence of variable selection have not been studied sufficiently in the literature.

In this paper, we contribute to the literature by studying three aspects of learning with selection of forecasting rules under attention costs.[2] First, we extend the standard AL setup by incorporating a dynamic dimensionality reduction, realized as a costly attention allocation problem. This makes the number of variables used in the forecasting rule time-varying and make the learning algorithm non-smooth. This non-smoothness could give rise to the sliding dynamics that appear along the boundary where more than one forecasting rule is optimal. We believe that a description of sliding dynamics in the adaptive learning literature is novel. Second, we contribute to the AL literature by addressing the global stability of the Minimum State Variable (MSV) Rational Expectations Equilibrium (REE) taking into account this non-smooth learning algorithm, where the agents start in a Restrictive Perception Equilibrium (RPE) with a forecasting rule that differs from MSV REE, and can switch the rule during learning. This RPE can be thought of as an outcome of an AL procedure with zero attention costs, when the agents are allowed to take only one variable into account. However, after such a convergence, if the agents could include the second variable into their forecasting rule subject to attention costs, and continue learning, they will change their rules and converge to the MSV REE. Thus, global E-Stability of the MSV REE can be envisioned in a very large region of initial beliefs. The RPE can be considered an artifact of adaptive learning under a very strong information exclusion constraint. Third, we study the role of monetary policy for dynamics and the equilibrium choice of forecasting rules and anchoring of inflation expectations. Although we study a model with only a few variables, the intuition we develop regarding model selection can be extended to large scale models with multiple variables or groups of variables.

To address these aspects, we study the dynamics of a model with boundedly rational agents, who operate under a combination of two types of information constraints: Recursive Least Squares (RLS) learning[3], with the agents updating their beliefs about the coefficients in their forecasting rules, and Sparse Rationality, which imposes costs on the attention weights of different variables in the fore-

[1] Examples include principal components analysis: Stock and Watson (2002), dynamic factor models: Banbura et al. (2013), Stock and Watson (2016), and active spaces or penalized regressions: see Hansen and Liao (2019), Korobilis (2013), and Nazemi and Fabozzi (2018). A systematic way of using many variables in econometric and forecasting models is represented by many variants of penalized regressions, including Ridge, Lasso, elastic nets, etc.: see Gefang (2014), Tibshirani (1996), De Mol et al. (2008), or Yuan and Lin (2007). See also Andrle and Bruha (2023) for sparse Kalman filter estimation.

[2] We are using terms "forecasting models" and "forecasting rules" interchangeably unless there is a reason to distinguish them.

[3] A variety of other variants is possible, such as Constant Gain learning, Kalman Filter learning, etc.

casting rule, thus selecting variables to be used in the rule. We call this algorithm Sparse Adaptive Learning (Sparse AL). The combination of these two concepts of bounded rationality allows us to study dynamic model selection with costly attention for a variety of initial beliefs, including when agents start with mis-specified forecasting rules.

Our agents use their estimates of the forecasting rules to form expectations of the future values of macroeconomic variables and agent actions that affect future realizations of data. The process of expectations formation then becomes self-referential. In Sparse AL, the agents are not only updating the regression estimates, but are also constantly deciding on the amount of attention they will pay to different variables, thus adding another mechanism into the usual self-referential feedback loop studied in the adaptive learning literature.

The Sparse Rationality approach formulated in Gabaix (2014) is a type of penalized regression, specifically a non-negative garrote. A penalized regression minimizes the loss function that consists of the sum of squared prediction errors plus a penalty term. In garrote, the penalty is imposed on the sum of absolute values of *attention weights* on different variables. Solving the problem of minimizing the loss function results in the derivation of optimal attention weights, some of which could be zero due to the functional form of the penalty, thus inducing sparsity. The choice of garrote estimator for a dimensionality reduction is motivated by its forecasting performance combined with its ability to preserve the story-telling properties of the model. In macroeconomic modelling, it is important to understand the economic logic behind predictions and, even more importantly, to be able to communicate the results to policy makers. Some classical dimensionality reduction methods clearly lack story-telling properties, while garrote allows us to study agents making choices among models that support economic narrative.

We are interested in the dynamics of the model that starts with initial beliefs that are significantly different from those consistent with the Minimum State Variable Rational Expectations (MSV REE) equilibrium developed by McCallum (1983, 2003). In particular, motivated by Audzei and Slobodyan (2022), we allow our agents to start arbitrarily close to the “wrong" RPE, in which case the initial forecasting rule is over-parametrized and includes a variable that is absent from the MSV REE solution.

We study the approximating dynamics of our learning algorithm represented by the solutions to a system of ordinary differential equations (ODEs). This approach allows us to more precisely characterize the convergence properties and dynamics across the areas where different forecasting rules are optimal. Our analysis of sliding dynamics is enabled by our reliance on continuous time ODEs.

We find that the dynamics under under Sparse AL generally converge to the MSV REE-consistent forecasting rule, even if the agents' initial beliefs are consistent with the RPE and are far away from MSV consistent beliefs. Because this convergence happens for initial beliefs that are not located in a small neighborhood of the MSV REE, we can speak about the global E-Stability of the MSV REE. The area in the parameter space consistent with convergence to the MSV REE forecasting rule under Sparse AL learning dynamics with attention costs constraints is significantly larger than in Audzei and Slobodyan (2022). Moreover, the RPE in our model, which is E-stable conditional on the limited information set, is unstable when a richer information set is introduced, and does not survive in the long run under Sparse AL. This suggests that the very existence of the RPE could be a fragile event that is caused by conditioning on the limited information available to the agents. Further, for larger values of the attention costs, the forecasting rule that contains only the constant can become stable, anchoring the inflation expectations to that constant. We call this an anchored rule.

A stronger monetary policy reaction to inflation facilitates the model's transition from an RPE towards an MSV-consistent rule. A stronger monetary policy reaction decreases the feedback from expectations and limits the possibility of a "wrong" forecast becoming self-fulfilling. Thus, the agents switch to include a "correct" variable and drop the "wrong" variable sooner, in a shorter time and thus a smaller number of iterations.

A more aggressive monetary policy somewhat expands the area in which an anchored rule is a stable equilibrium, consistently with Audzei and Slobodyan (2022). This happens because a stronger monetary policy reaction to inflation reduces the volatility and persistence of inflation in the model economy. When attention is costly and the volatility and predictability of the variables in question is low, the agents are less willing to pay costly attention to learn about their dynamics. Instead, agents utilize an anchored rule so that their inflation expectations are anchored at the long-term inflation average.[4]

1.2 Literature Review

This paper is related to a literature on the survival of mis-specified equilibra in self-referential systems. Evans et al. (2012) showed that the convergence to a mis-specified equilibrium occurs when the feedback parameter on the expectations is strong. A similar conclusion was obtained by Adam (2005), Hommes and Zhu (2014), Hommes (2014), Branch et al. (2022) and Hajdini, but under different formulations of mis-specification and learning processes. Our main contribution to this literature is adding variable selection and sparsity considerations as in Gabaix (2014) to Adaptive Learning, and studying the global (Sparse) E-stability of the resulting dynamic system. We find that asymptotic stability of a particular mis-specified equilibrium (the RPE) can be very fragile, and that the RPE would not survive after allowing the agents to consider *combinations* of mis- and well-specified rules, instead of comparing the performance of a fixed set of rules, as is typically done in this literature. As in Audzei and Slobodyan (2022), we also show that a stronger monetary policy response to inflation, which is inversely related to the expectational feedback parameter, makes the survival of a mis-specified equilibrium less likely.

In our framework, agents' dynamic decisions on including or excluding the variables from the forecasting rules introduces discontinuity into the model dynamics when the set and number of included variables changes. This potential discontinuity forces us to rely on the theory of non-smooth differential equations, (see Filippov (1988) and Jeffrey (2019)), for studying the approximating behavior of the Sparse AL algorithm. In addition to the standard convergence of learning dynamics, *sliding* dynamics along the boundary where the agents are indifferent between two different forecasting rules can be observed. The appearance of sliding affects the convergence properties of different equilibria.

Further, we refer to a large strand of literature on AL and its interaction with monetary policy. Summaries of the AL approach are provided in Evans and Honkapohja (2001) and Marcet and Sargent (1989). Seminal contributions related to AL interaction with monetary policy include Orphanides and Williams (2007), who studied the robustness of monetary policy rules when agents are learning.

[4] In case of inflation, this long term average may or may not coincide with the inflation target, therefore for the purpose of this paper we abstract from normative statements about desirability of such an outcome for a central bank.

The monetary policy analysis in our paper is more related to the studies that address how monetary policy affects the learnability and stability of the equilibria under the learning process: see Mele et al. (2020), Bullard and Mitra (2002), Slobodyan et al. (2016), Christev and Slobodyan (2014) and Gibbs (2017). In these studies, the learnability and stability of a desired equilibrium is viewed as additional desiderata for a monetary policy rule. We distinguish this work from this literature by considering an environment with costly attention. Large attention costs alter monetary policy impact on expectation formation, as stabilizing inflation and output results in lower agent incentives to learn about them.

Our research question is also related to the literature on model validation: Cho and Kasa (2015) study validation of alternative models based on large deviations theory. In contrast to large deviation theory, we study only the mean or average, dynamics of the model rather than the probabilities of tail events. An important difference between our analysis and that of Cho and Kasa (2015) is that one of the forecasting rules is MSV consistent, which allows our agents to learn the true model of the world, while in Cho and Kasa (2015), all models are potentially mis-specified.

Our formulation of the initial mis-specified equilibrium is inspired by empirical and theoretical literature on RPEs. Studies have demonstrated that models with simple prediction rules for inflation outperform those applying complicated rules in survey and experimental settings: see Branch and Evans (2006), Adam (2007), Hommes (2014), and Pfajfar and Žakelj (2014). Survey literature supports the existence of different mental models: For example, Jiang et al. (2024) finds that most consumers do not adjust their consumption in the short run to changes in inflation expectations. Heterogeneity in macroeconomic forecasts due to emphasis on different transmission mechanisms is highlighted in Andre et al. (2022). In relation to the models' behavior at the effective lower bound, Ascari et al. (2023) show that combining the RPE and bounded rationality helps to restore the uniqueness of an equilibrium. In the context of estimated New Keynesian models, Slobodyan and Wouters (2012a, 2012b), Audzei (2023), Ormeno and Molnar (2015) and Vázquez and Aguilar (2021), have shown that assuming that agents use very simple forecasting rules leads to model fit in estimated DSGE models under adaptive learning that is superior to that under RE. Hajdini shows that simple rules are consistent with consensus inflation forecasts. While Hajdini combines mis-specified forecasting rules with myopia à la Gabaix (2020b), we combine mis-specified rules with costly attention as in Gabaix (2014), such that agents' choices depend on the volatility and persistence of the variables given attention costs. This type of formulation changes model predictions for some range of attention costs and variable predictability, when it becomes optimal for the agents to use only a constant as a predictor for inflation.

The paper is structured as follows. We start by describing the concept of Sparse AL and the global stability of the models in this context. We illustrate global convergence under Sparse AL concept in a simple framework of learning about the Fisher equation in Section 3. In this framework, we are able to study the transition of models between equilibria and the role of monetary policy in the evolution of forecasting models. We continue with a small-scale general equilibrium NK model with expert advice in Section 4. We further study the stability and learnability of different equilibria in Section 5, where we also address the global E-stability of MSV REE using the theory of non-smooth differential equations. We demonstrate analytically how the non-smooth dynamics allow us to compare the model dynamics under fast and slow updating of underlying variable selection. The final Section concludes.

2. Sparse Rationality under Adaptive Learning

The Sparse Rationality concept, introduced by Gabaix (2014, 2020a) considers the following decision problem. Suppose an agent wants to minimize a loss function associated with some costly action. If the loss function is given as a sum of squares of forecast errors from a linear forecasting problem plus the cost of action, which is a function of the regression coefficients of this regression, the problem is that of *penalized regression*. The penalty term can be interpreted as the sum of the costs of paying attention to the variables due to information or data collection efforts.

Under Gabaix's concept, agents choose the weights of attention to the variables depending on their importance in decision making, where *importance* depends on the predictors' contribution to the variance of the predicted variable and the utility of agents. In the context of the forecasting process, one could interpret the weight of attention as the frequency of data revisions, recalibration of trends, and/or of steady states. If the weight on a variable is zero, agents use its default or steady state value. In a model linearized around the steady state, zero attention weight means that the agents do not include the variable in their regressions. When the attention weight is unity, it means that the forecast is fully adjusted for deviation of the variable from the steady state value. Selecting attention weights equal to zero or one is thus equivalent to a classical model selection exercise. For example, when energy prices rise, central banks can ignore their deviation from the steady state when the deviation is considered to be temporary and with little propagation to quarterly inflation. Thus, the weight of energy prices in an inflation forecast will be zero. At the other extreme, when a central bank estimates that energy prices are the major driver of inflation, the weight on energy prices in the forecast is unity. An attention weight between zero and unity reflects the degree by which a deviation in energy prices is incorporated into inflation forecasts.

Formally, the decision problem is formulated as follows. Suppose that the agents populate an economy that is described by a general expectational difference equation with matrices of coefficients A, B, C:

$$y_t = AE_t y_{t+1} + BE_t y_{t-1} + Cu_t. \tag{1}$$

Note that the agents' expectations about future inflation affect the actual dynamics of inflation. The choice of forecasting model influences the dynamics of the model through the expectational term.

The agents form their expectations by using a linear forecasting rule - a perceived law of motion (PLM). They obtain the regression coefficients $\boldsymbol{\beta}$ in their forecasting rules by applying the usual OLS regression or a linear projection of the response variable y on a set of regressors $\boldsymbol{x}$:

$$y_t = \beta_1 x_{1,t-1} + \ldots + \beta_n x_{n,t-1} + \varepsilon_t = (\boldsymbol{\beta}_{t-1} \odot \boldsymbol{m}_{t-1})^T \boldsymbol{x}_{t-1} + \varepsilon_t, \tag{2}$$

where $\boldsymbol{\beta}$, $\boldsymbol{m}$ and $\boldsymbol{x}$ are vectors of beliefs, weights, and regressors, respectively.

They then form forecasts as $\hat{y}_t = \beta_1 m_1 x_{1,t-1} + \ldots + \beta_n m_n x_{n,t-1}$, with m_i being the *attention weight* allocated to the variable x_i. The agents then maximize the quality of their forecast: $u = -\frac{1}{2}E\left[(\hat{y}_t - y_t)^2\right]$, subject to attention costs $\kappa \sum_{i=1\ldots n} |m_i|$, where κ is the attention costs parameter. The optimal attention vector $\boldsymbol{m}$ is thus obtained as a solution to the following problem:

$$\boldsymbol{m}^* = \arg \min_{m\in[0,1]^n} \left\{ \frac{1}{2}E\left[(\hat{y}_t - y_t)^2\right] + \kappa \sum_{i=1\ldots n} |m_i| \right\} \equiv G(\boldsymbol{\beta}_{t-1}, \boldsymbol{m}_{t-1}). \tag{3}$$

The problem (3) is known as a (non-negative) garrote, and the weights can take any value between 0 and 1. Specifying the penalty term as the sum of absolute values ensures that the corner solutions with attention weights of 0 and 1 are possible. Minimizing the loss function with an attention cost penalty is then akin to running a classic model selection exercise; however, it is also possible to pay partial attention to a variable when $0 < m_i < 1$.

We now allow the agents to continue learning adaptively, taking into account the attention cost. As is usual in the adaptive learning literature, they run Recursive Least Squares (RLS) to adjust the values of beliefs $\boldsymbol{\beta}$ and the second moments of the explanatory variables R, according to equations (4-5):

$$\begin{aligned} \boldsymbol{\beta}_t = \boldsymbol{\beta}_{t-1} + & \\ + t^{-1} \cdot R_t^{-1} \cdot \boldsymbol{x}_{t-1} \cdot (T(\boldsymbol{\beta}_{t-1} \odot \boldsymbol{m}_{t-1})^T \cdot \boldsymbol{x}_{t-1} + \varepsilon_t - (\boldsymbol{\beta}_{t-1} \odot \boldsymbol{m}_{t-1})^T \cdot \boldsymbol{x}_{t-1})^T, & \quad (4) \\ R_t = R_{t-1} + t^{-1} \cdot (\boldsymbol{x}_{t-1} \boldsymbol{x}_{t-1}^T - R_{t-1}). & \quad (5) \end{aligned}$$

However, in addition to the standard RLS, the agents also recompute the optimal attention weights $\boldsymbol{m}^*$ given $\boldsymbol{\beta}_{t-1}$ and $\boldsymbol{m}_{t-1}$, and adjust their current weights $\boldsymbol{m}$ towards the optimal values, as in equations (6)-(7):[5]

$$\boldsymbol{m}_t = \boldsymbol{m}_{t-1} + nt^{-1} \cdot (\boldsymbol{m}_t^* - \boldsymbol{m}_{t-1}), \quad (6)$$
$$\boldsymbol{m}_t^* = G\left(\boldsymbol{\beta}_{t-1}, \boldsymbol{m}_{t-1}\right). \quad (7)$$

We call this combination of a usual Adaptive Learning and Sparse Rationality a *Sparse AL*.

2.1 Approximating the Dynamics of Sparse AL

While the algorithm in (4)-(7) could be studied in its original discrete-time form, we opt to investigate its approximating average dynamics, which is a multi dimensional nonlinear ODEs. While we show most numerical simulations for the models in Sections 3-4 using the algorithm in (4)-(7), working with the approximating ODE is very helpful for analyzing the dynamics of the simple model in Section 3 around the RPE and sliding dynamics for the New Keynesian model in Section 5.2. Further, we obtain E-Stability results using approximating ODE.

The approximating ODE for the described learning algorithm is given by the equations (8) below:

$$\begin{aligned} \dot{\boldsymbol{\beta}} &= T(\boldsymbol{\beta} \odot \boldsymbol{m}) - \boldsymbol{\beta} \odot \boldsymbol{m}, \\ \dot{R} &= \Sigma - R, \\ \dot{\boldsymbol{m}} &= n \cdot (G(\boldsymbol{\beta}, \boldsymbol{m}) - \boldsymbol{m}). \end{aligned} \quad (8)$$

Following Evans and Honkapohja (2001), section 6.2.2, one can interpret this ODE as the agents fixing the parameters of their forecasting rules - $\boldsymbol{\beta}$, R, and $\boldsymbol{m}$ - and observing the terms, multiplied by the gain, on the right-hand side of (4)-(7), for a very large number of periods. Then, they average these right-hand side terms, and make an infinitesimal step in the direction of the average value. They also make an infinitesimal step towards the attention weights that would be optimal in these periods.

[5] Note that we allow for difference in the gain in beliefs updating equations (4)-(5), $1/t$, and in the weights updating equation (6), n/t.

Fixing $n = 1$ in (6) then implies that the agents adjust their beliefs, $\boldsymbol{\beta}$ and R, as often as they adjust the attention weights $\boldsymbol{m}$, while $n < 1$ signifies a less frequent adjustment of attention weights than of beliefs. Given that solving an OLS problem is much simpler than obtaining the solution to (3), which in general requires usage of advanced convex optimization tools such as LARS, it is natural to assume that the agents would update weights less often than beliefs and thus use $n \ll 1$. In our numerical simulations later in the text we compare the results for $n = 1$ and $n = 0.01$.

2.2 Global Convergence and Initial Beliefs

When the agents are using the MSV functional form to formulate their PLM and then use the RLS learning algorithm to learn the coefficients in their PLM, the dynamics of their currently assumed coefficients (beliefs) is asymptotically governed by the approximating ODE. Asymptotic stability of a stationary point of this approximating ODE, or weak E-stability, is related to the convergence of the agents' beliefs to their MSV REE values: see Evans and Honkapohja (2001) and Marcet and Sargent (1989).

In this paper, we are mainly interested in a form of the *global strong* E-stability concept of the MSV solution. The *weak* E-stability guarantees only that if the agents' forecasting rule has the same functional form as the MSV solution, and the values of coefficients they believe in are initially contained in some small neighborhood of the MSV values, then under appropriate assumptions, the RLS learning will converge to MSV REE with a probability approaching 1; see Evans and Honkapohja (2001). This result leaves two unanswered questions. First, what happens if the initial forecasting rule contains more variables than the MSV-consistent rule? Will the agents asymptotically learn that the values of extraneous coefficients in the PLM are zero? In other words, does the *strong* E-stability obtain? Second, what happens if the initial beliefs are far away from the MSV ones: will we still observe convergence? That is to say, is the weak or strong E-stability not just *local (asymptotic)*, but *global*?

Unlike the weak E-stability, a strong E-stability concept is not unique. It is defined only with respect to the specific mis-specification of the PLM from which the learning is assumed to begin. To answer the first question, we derive the conditions under which the strong E-stability obtains when the agents allow an additional endogenous variable to be present in their PLM for the two models we consider in this paper. We show that if the sufficient condition for the weak E-stability is satisfied, strong E-stability obtains as well. With strong E-stability, the beliefs will still converge to the correct MSV REE beliefs and the agents will learn that the additional variable does not belong in their PLM; this is guaranteed to occur when the initial beliefs are sufficiently close to the MSV REE ones, and thus the initial beliefs about the additional variable are close to zero.

In order to answer the second question, we consider agents' initial beliefs that are as far away from the MSV as possible, while still resembling the MSV functional form. To do so, in this paper we start with the beliefs that are consistent with the Restricted Perceptions Equilibrium. That is, with all agents using a forecasting rule that includes some "wrong" variables that are absent from the MSV and/or that exclude "correct" ones. When the agents use the 'incorrect' set of variables for forecasting, due to the self-referential nature of the beliefs, the resulting Restricted Perceptions Equilibrium is skewed, and it could happen that the forecasting errors are smaller than they would have been had the agents used the 'correct' variables in this skewed RPE.

We start with a simple self-referential univariate model and then move to a New-Keynesian model with a richer structure and standard set of frictions.

3. Simple Model: Fisher Equation

To illustrate the dynamics of model selection under Sparse AL in a simple example with analytical results, we employ a Fisher equation with an exogenous real interest rate. The nominal interest rate is given by a simple Taylor rule with a zero inflation target. The model can be summarized as follows:

$$R_t = r_t + E_t\pi_{t+1}, \tag{9}$$
$$r_t = \rho r_{t-1} + u_t, \tag{10}$$
$$R_t = \phi\pi_t, \tag{11}$$

where r_t is the real interest rate and π_t is inflation. Eq. (9) is the Fisher equation with the nominal interest rate R_t set according to the Taylor rule (11), and the real interest rate in (10) is an AR(1) exogenous process with stochastic disturbance u_t. Parameter ϕ controls the strength of monetary policy reaction to inflation. The agents observe the current real rate r_t and inflation from the prior period π_{t-1} while making forecasts for inflation in the next period.

Plugging (11) into (9) we obtain the dynamics of inflation:

$$\pi_t = \frac{1}{\phi}r_t + \frac{1}{\phi}E_t\pi_{t+1}, \tag{12}$$

where the term $1/\phi$ determines how strongly the expectations affect the actual dynamics of inflation: The higher this term is, the stronger the effect of agents' forecasting rule on actual inflation.

To form their forecasts, our agents run regressions using one or both observables - the current real interest rate or lagged inflation. In general, the agents can use a PLM of the following form:

$$\pi_t = \gamma r_t + \beta\pi_{t-1}, \tag{13}$$

where γ and β are their beliefs on r_t and π_{t-1} respectively. Note that the rule (13) nests cases in which only one variable is used for forecasting ($\gamma = 0$ or $\beta = 0$), or where none of the variables are used ($\gamma = \beta = 0$).

Computing agent expectations and plugging them into (12), we receive the actual law of motion - ALM:

$$\pi_t \quad = \frac{1}{\phi}r_t + \frac{1}{\phi}\Big(\gamma(\beta+\rho)\,r_t + \beta^2\pi_{t-1}\Big) = \bar{c}r_t + \bar{b}\pi_{t-1}, \tag{14}$$

with $\bar{c} = \frac{1}{\phi}(\gamma(\beta+\rho)+1)$ and $\bar{b} = \frac{1}{\phi}\beta^2$ being the ALM coefficients on r_t and π_{t-1}.

In Appendix A.1 we derive the REE MSV and show that sufficient conditions for strong E-stability coincide with sufficient conditions for weak E-Stability. We are interested in the dynamics of model selection when the agents start with a “wrong" forecasting rule. Motivated by empirical and theoretical work on restricted perception, we allow our agents to use only one variable in their initial PLM. There are two such rules:

$$M_\pi \quad : \pi_t = \beta\pi_{t-1} \Rightarrow E_t\pi_{t+1} = \beta^2\pi_{t-1}, \tag{15}$$
$$M_r \quad : \pi_t = \gamma r_t \Rightarrow E_t\pi_{t+1} = \gamma\rho r_t. \tag{16}$$

While M_r is the MSV-consistent forecasting rule, M_π is the “wrong", mis-specified rule as it omits the state variable r_t and contains the “wrong" extra variable π_{t-1}. We also call the equilibria induced by these forecasting rules M_π and M_y respectively. M_π is the RPE.

M_π could exist if agents' perceived autocorrelation of inflation equals its true value obtained from projecting π_t on π_{t-1}. As shown in the Appendix A, the equilibrium β is given by the unique real solution of (17):

$$\beta = \frac{cov(\pi_t, \pi_{t-1})}{var(\pi)} \equiv \Gamma(\beta). \tag{17}$$

We list other conditions for RPE to be an equilibrium in Definition 1.

Definition 1. *The RPE M_π exists if all of the following conditions are satisfied:*
(i): There exists β s.t. $|\beta| < 1$, which is a fixed point of mapping Γ in (17);
(ii): the ex-post *forecasting performance of M_π is better than that of M_r;*
(iii) M_π equilibrium is E-stable.

In the RPE, the forecasting performance of the rule M_π measured by the mean squared forecast errors (MSFE) should be better than that of M_r. Appendix A shows that for the rule M_π to have lower MSFE than M_r, the variation explained by the lag of inflation should be larger than that explained by the real interest rate:

$$\bar{b}^2\sigma_\pi^2 > \bar{c}^2\sigma_r^2. \tag{18}$$

For condition (iii) to be satisfied, that is, for the solution to be E-Stable, the following should hold:

$$\frac{\partial\Gamma(\beta)}{\partial\beta} < 1. \tag{19}$$

Proposition 1 states the conditions under which this mis-specified rule becomes an equilibrium.

Proposition 1. *The necessary condition for (i)-(iii) to be satisfied is the Taylor principle $\phi > 1$, the sufficient condition is $\frac{\phi^2-2\beta(\rho,\phi)^4}{\phi\beta(\rho,\phi)^2} < \rho < 1$ plus the Taylor principle.*

Proof. Appendix A.2. □

The Taylor principle is a usual necessary condition for E-stability in monetary models. As higher persistence of the real interest rate results in greater correlation with inflation, a larger portion of the actual variation of the forecast variable (inflation) is explained by the “wrong" variable (lagged inflation), making M_π perform better than M_r.

Further, for a more aggressive monetary policy, the explanatory power of M_π becomes too small to produce better forecasts than M_r. As ϕ becomes larger, the expectational feedback in Eq. 12 becomes smaller, lowering the contribution of the mis-specifed rule to the dynamics of inflation.

3.1 Dynamics Under Sparse Rationality

Suppose that the parameters of the model are such that the RPE exists and is given by the corresponding ALM in (14). We allow the agents to have initial beliefs (β, γ, R) consistent with the RPE. Their initial attention weights are $(m_\pi, m_r) = (1, 0)$. The agents then dynamically reconsider their forecasting rules in line with Sparse AL described in Section 2. They update their beliefs about (β, γ, R) according to the standard adaptive learning procedure (4)-(5), solve the optimal weight selection problem by computing the function $G(\boldsymbol{\beta}, \boldsymbol{m})$ in (3), and update the weights as in (6). The resulting dynamics are approximated by the system of ODEs (8). Analyzing the RHS of (8) under the assumption that agents' beliefs about second moments R are also at their equilibrium values, we state the Proposition 2:

Proposition 2. *Under Sparse AL, the agents' beliefs about inflation are always decreasing as* $\frac{\partial \beta}{\partial \tau} < 0$. *Their beliefs about the interest rate are non-decreasing as* $\frac{\partial \gamma}{\partial \tau} > 0$.

Proof. Appendix A.3. □

Proposition 2 states that, as we start from the RPE beliefs with $(m_\pi, m_r) = (1, 0)$ and $\beta < 1$, β is decreasing as the agents learn. As weight on inflation lag m_π cannot increase above unity, $m_\pi \beta$ also falls. On the other hand, the belief about interest rate γ increases. The value $m_r \gamma$ initially stays at zero as the optimal m_r^* is 0 at the RPE.

In other words, as soon as we allow the agents to move along two dimensions rather than one, the dynamics is to decrease the beliefs about inflation and to increase them about the interest rate. As shown in the Appendix A.3, this behavior is likely to be preserved even if the agents are mistaken about the second moments matrix R implied by the RPE ALM, but their beliefs about it are not too far from its true value.

Thus, the RPE is a stable fixed point in 1D space, as the condition (iii) of Definition 1 is satisfied, but it is not a stable fixed point in the 2D space. This is explained by the difference between the RPE's approximating ODE and the first line of the 2D approximating ODE in (A15). The reason for this is that, in the 1D case, any movement away from the RPE value of β leads to changes in correlation between inflation and interest rate that need to be taken into account, while in the 2D case these dynamics can be 'moved' into the second dimension. This means that the RPE value of β **is not** the first component of the stationary point in the 2D case, and therefore immediately after opening the second dimension, we have the dynamics of learning moving agents away from the RPE beliefs.

Asymptotically as $\beta \to 0$ in (A15), the ODE for β becomes linear and has a fixed point $\beta = 0$. Therefore, Sparse AL dynamics can reach the point at which agents take no account of inflation in their PLM. But what will their final forecasting rules be? There are two model candidates that pay no attention to inflation: the anchored (constant only) rule and the MSV-consistent rule. When the weights become such that a set of variables in a forecasting rule changes in the updating process, we call it a model switch. Therefore, we now consider the sequence of switches that could lead the agents from the attention weights with $(m_\pi > 0, m_r = 0)$ to $(m_\pi = 0, m_r \geq 0)$.

3.2 Forecasting Model Switches and Sparse Rationality

Under sparse rationality, the agents' forecast depends not only on the underlying beliefs β and γ, but also on the selected weights. The expectations are formed using these beliefs and the weights

m_π and m_r, which produce the ALM:[6]

$$\pi_t \quad = \frac{1}{\phi}\left(m_r\gamma(m_\pi\beta+\rho)+1\right)r_t + \frac{(\beta m_\pi)^2}{\phi}\pi_{t-1} = \bar{c}r_t + \bar{b}\pi_{t-1}. \tag{20}$$

The agents use a standard AL procedure to update their beliefs. On the average, this is given by the first line of the ODE (8). In order to find optimal weights under sparse rationality, they then standardize the variables and compute the modified ALM coefficients:

$$\bar{b}^s \quad = \bar{b}\frac{\sigma_\pi}{\sigma_\pi} = \bar{b}, \tag{21}$$

$$\bar{c}_s \quad = \bar{c}\frac{\sigma_r}{\sigma_\pi} = \sqrt{\frac{(1-\rho\bar{b})(1-\bar{b}^2)}{1+\rho\bar{b}}}. \tag{22}$$

Note that the standardized value of $\bar{c}_s$ in (22) is independent of $\bar{c}$ and thus of γ, leaving only $\bar{b} = \frac{(m_\pi\beta)^2}{\phi}$ as the relevant PLM belief for the purposes of computing forecasts in the attention weights problem. This is due to the fact that the true state variable of the model, r_t, is exogenous, and its variance is thus independent of PLM beliefs. Moreover, in this setup, $\bar{c}_s$ is decreasing in $\bar{b}$, so that while $m_\pi\beta$ is decreasing while the agents learn, $\bar{c}_s$ increases. This leads to very simple sequencing of optimal model switches, as we show below.

Solving for the weights in (3) is equivalent to selecting the lowest of the nine value functions (term in the figure brackets in Eq. 3), which are derived in Appendix A, Eqs. A20-A27. This determines the forecasting model that is optimal for particular values of the agents' beliefs and given the parameter values. To further understand the dynamics of model selection, we therefore compute the boundaries between different value functions being optimal, which informs us about the order in which the agents switch between different forecasting models. As only the product of their beliefs and the weight on **inflation** is relevant for the forecasts, expected forecast errors, and thus the weight selection problem, for particular parameter values these boundaries will be points in a 1-dimensional space.

The result of selecting the optimal weights is illustrated in Figure 1. The Figure shows agents' selection of attention weights as a function of their current value of $m_\pi\beta$ and of the attention cost parameter κ. The arrows illustrate the dynamic trajectories of the agents' beliefs as well as the outcomes of the model selection. While Figure 1 is plotted for a specific value of ρ and ϕ, below we provide a general characterization of the agents' decision to switch between the optimal models.

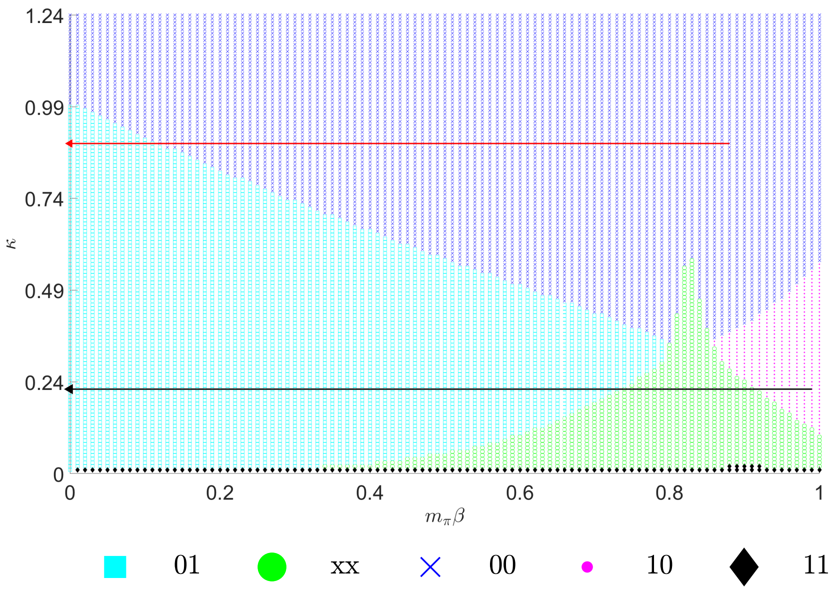
Depending on the value of costs κ, the trajectory of beliefs towards $\beta = 0$ crosses several boundaries between the regions where a particular combination of weights is optimal, summarized in the following Proposition:

Proposition 3. *There exist boundaries $b_1 < b_2$, such that for low values of κ:*

- *for $m_\pi\beta > b_2$ agents use the RPE-consistent rule, $(m_\pi, m_r) = (1,0)$;*
- *for $b_1 < m_\pi\beta \leq b_2$ both attention weights are positive, $(m_\pi > 0, m_r > 0)$;*

[6] Note that the Eq. (14) is a special case of this expression for $m_\pi = m_r = 1$.

Figure 1: Dynamic Model Selection and Costs of Attention



Note: For the Figure we use numerical values of $\rho = 0.5$ and $\phi = 1.15$. The black arrow illustrates dynamics from RPE to MSV, the red arrow - from (0,0) to MSV.

- *for $m_\pi\beta \le b_1$ the agents choose the MSV-consistent rule, $(m_\pi, m_r) = (0,1)$.*

For large values of κ there exists one border b_3, such that if

- *for $m_\pi\beta > b_3$, the agents choose the anchored rule,*
- *for $m_\pi\beta \le b_3$, the agents choose the MSV-consistent rule*

There exists $\bar{\kappa}$ such that $b_3 \le 0$. For $\kappa \ge \bar{\kappa}$, only the anchored rule is possible.

Proof. Appendix A.4 □

Proposition 3 states that, as we move from a pure RPE with the weights $(m_\pi, m_r) = (1,0)$ along the path shown as a black arrow, the agent beliefs first hit the boundary b_2 between the area with RPE consistent beliefs $(m_\pi, m_r) = (1,0)$ being optimal, and the area where there are positive weights on both variables $(m_\pi, m_r) = (x,x)$. When $m_\pi\beta \le b_2$, it becomes optimal for the agents to allow a nonzero weight on the real interest rate. The actual value of $m_r\gamma$ starts to increase above 0 after crossing boundary b_2, in accordance with (8).

As agents add the real interest rate to their regressions, $m_\pi\beta$ continues to decrease, eventually becoming such that $\bar{b}^2 < \kappa$. This is the boundary with the region where including the lag of inflation into the forecasting rule is not optimal. In this region, m_π and $m_\pi\beta$ decrease towards 0 while $m_r\gamma$ converges to MSV-consistent value.

Proposition 3 further states that, when attention costs κ are large (the trajectory of beliefs shown as a red arrow in Figure 1), for large initial beliefs about $m_\pi\beta$ it is optimal to include no variables in the forecasting rule, $(m_\pi^* = m_r^* = 0)$. According to Eq. 8, m_π starts to decline while β falls as per Proposition 2. With $m_\pi\beta$ declining, the explanatory power of the real interest rate grows because $\bar{c}_s$ is a decreasing function of $m_\pi\beta$ as noted above. Eventually, the trajectory hits the boundary $m_\pi\beta = b_3$, where the MSV-consistent forecasting rule starts to be optimal, so that convergence to the MSV beliefs ensues.

For still higher attention cost κ the MSV-consistent beliefs never become optimal, and the agents stick with the anchored rule forever $(0,0)$.

When costs of attention κ are zero, standard adaptive learning dynamics ensue, and the beliefs converge to the MSV, which is weakly and strongly E-stable as long as $\phi > 1$, as shown in the Appendix A.1.

Finally, for intermediate values of κ one can observe additional boundaries, for example between $(m_\pi, m_r) = (0,0)$ and $(m_\pi, m_r) = (x,x)$. Convergence to the MSV is still the ultimate outcome of the Sparse AL algorithm.

3.3 Monetary Policy and Discussion

We next comment on the effect the monetary policy has on the convergence of the agents' beliefs to the rest point, where inflation is not taken into account.

Proposition 4. *As monetary policy reaction to inflation becomes stronger (ϕ increases), the speed of convergence of beliefs to $m_\pi\beta = 0$ increases.*

Proof. From (A15) and (A16) it follows that $\frac{d\beta}{d\tau}$ is decreasing in ϕ. Therefore the product $m_\pi\beta = 0$ declines faster, and thus the speed of convergence to the fixed point with $m_\pi\beta = 0$ is greater. □

Proposition 4 states that the effect of strong monetary policy reaction to inflation on the dynamics of Sparse AL is unambiguous. The convergence from any initial beliefs, including the RPE-consistent ones, to an equilibrium where inflation does not matter for the agents' forecasting, is accelerated by the strength of monetary policy: in this simple model, a more aggressive monetary policy results in the agents' learning more quickly that the wrong variable does not belong to their forecasting rules.

Our simple model analysis allows us to show the following findings regarding the dynamics of model selection. First, we have shown how agents' forecasting models converge to the MSV-consistent one even if they start with beliefs far from the MSV, with initial RPE-consistent PLM being 'orthogonal' to the MSV. In other words, the dynamics confirm global strong E-stability of the MSV REE in the model. The existence of the RPE is thus an artifact that is caused by artificially restricting the agents' information set. For very large attention costs, the agents asymptotically choose to use the anchored rule, which shows that, while the region of attraction of the MSV is large, it does not contain *all* possible parameter values.

Second, in line with the literature and our previous work in a static context Audzei and Slobodyan (2022), a stronger monetary policy reaction to inflation reduces the feedback of expectations to the realized variables. In the dynamic context, even if we start in the RPE and then allow the agents to consider including the true state variable into their forecasting rule, stronger monetary policy results in a faster switch to the MSV-consistent rule. We further illustrate the role of attention costs. Intuitively, with larger attention costs, the agents stop paying attention to any variables and start using the anchored rule. Nevertheless, they stop paying attention to the wrong variable first: the RPE-consistent rule disappears for $\kappa > 0.5$ while the MSV one does so for larger values of κ; see Figure 1.

In the next section, we study a more realistic text-book New Keynesian model with a richer structure, in which both possible variables in the PLM are endogenous. We show that the basic intuition about the instability of RPE under Sparse AL carries through even in this larger model. In addition, we investigate a novel phenomenon of sliding dynamics taking place between regions in which two different forecasting rules are optimal.

4. Full Model

To account for the role of monetary policy in expectation formation, we chose a standard New Keynesian (NK) model in which consumer utility possesses external habit persistence[7] and a central bank reacts to the deviation of expected inflation from the zero inflation target. The model has been studied extensively, including for monetary policy analysis; therefore we present below the key equations and leave the detailed derivations to the Appendix B. The model is a three equations NK model, with the investment - savings curve, new Keynesian Phillips curve and monetary policy Taylor-like rule:

$$y_t = -\frac{1-h}{(1+h)\sigma}(i_t - E_t\pi_{t+1}) + \frac{1}{1+h}E_t y_{t+1} + \frac{h}{1+h}y_{t-1} + g_t, \tag{23}$$

$$\pi_t = \beta E_t \pi_{t+1} + \omega y_t + u_t, \tag{24}$$

$$i_t = \phi_\pi E_t \pi_{t+1}. \tag{25}$$

Here π is inflation and y the output gap, while the shocks u and g are both *i.i.d.* zero mean random variables with finite variances.

We plug the central bank's policy rule (25) into (23)-(24) and rearrange to express the dynamics of inflation and output as a function of their lagged and expected values and shock realizations:

$$y_t = -\frac{1-h}{(1+h)\sigma}(\phi_\pi - 1)E_t\pi_{t+1} + \frac{1}{1+h}E_t y_{t+1} + \frac{h}{1+h}y_{t-1} + \frac{1-h}{(1+h)\sigma}g_t, \tag{26}$$

$$\begin{aligned}\pi_t = &\left(\beta - \frac{\omega(1-h)(\phi_\pi - 1)}{(1+h)\sigma}\right)E_t\pi_{t+1} + \\ &+ \frac{\omega}{1+h}E_t y_{t+1} + \frac{\omega h}{1+h}y_{t-1} + \frac{\omega(1-h)}{(1+h)\sigma}g_t + u_t.\end{aligned} \tag{27}$$

Next, we describe how agents formulate their inflation and output expectations.

4.1 Expert Forecasts of Output Gap

Following Molnar (2007), we assume that the agents have access to the expert advice on output gap forecasts. These experts are fully aware of the structure of the model in (26)-(27) and underlying parameters, and make forecasts *given* agents' inflation beliefs. We introduce the experts in order to reduce the dimensionality of the space of agents' beliefs, which allows us to generate some

[7] We want to study a framework where the RPE includes a lag of output gap. That is why we have chosen to consider a model with habit persistence.

analytical results in the two-dimensional model in Section 5. Without experts, our model becomes four-dimensional and unable to produce intuitive results.

In Appendix C we characterize the dynamics of output and inflation as a function of expectations of the output gap, given the agents' beliefs about inflation. As experts are fully rational, we solve for their output gap forecasts using the method of undetermined coefficients. As a result, the forecast of the output gap can be written as

$$E_t y_{t+1} = \tilde{\gamma}_y y_{t-1} + \tilde{\gamma}_\pi \pi_{t-1} + \tilde{\gamma}_u u_t + \tilde{\gamma}_g g_t, \tag{28}$$

where the coefficients are functions of the agents' inflation beliefs defined in (C9)-(C12) and (C13)-(C16).

With these output gap expectations, we can re-write the model in (26)-(27) as a function of inflation expectations and lags of inflation and output gap:

$$\begin{aligned} y_t = & -\frac{1-h}{(1+h)\sigma}(\phi_\pi - 1)E_t\pi_{t+1} + \frac{\tilde{\gamma}_\pi}{1+h}\pi_{t-1} + \\ & + \frac{h+\tilde{\gamma}_y}{1+h}y_{t-1} + \frac{1-h+\sigma\tilde{\gamma}_g}{(1+h)\sigma}g_t + \frac{\tilde{\gamma}_u}{1+h}u_t, \end{aligned} \tag{29}$$

$$\pi_t = \left(\beta - \frac{\omega(1-h)(\phi_\pi - 1)}{(1+h)\sigma}\right)E_t\pi_{t+1} + \tag{30}$$

$$\frac{\tilde{\gamma}_\pi\omega}{1+h}\pi_{t-1} + \frac{\omega(h+\tilde{\gamma}_y)}{1+h}y_{t-1} + \frac{\omega((1-h)+\sigma\tilde{\gamma}_g)}{(1+h)\sigma}g_t + \left(\frac{\omega\tilde{\gamma}_u}{1+h} + 1\right)u_t.$$

4.2 The Equilibria

4.2.1 Minimal State Variable Solution

Having defined the forecast of the output gap we begin our analysis by defining and studying the properties of MSV REE solution for inflation. At MSV, the agents' beliefs about the inflation - PLM, contain only the state variables: output gap and observed shocks. Such beliefs are given by:

$$\pi_t = c_\pi^y y_{t-1} + \gamma_\pi^y g_t + u_t, \tag{31}$$

$$y_t = c_y^y y_{t-1} + \gamma_y^y g_t, \tag{32}$$

with the coefficients derived in Appendix D.

4.2.2 Restricted Perceptions Equilibrium

Next we derive the Restricted Perceptions Equilibrium, where the 'correct' variable from the MSV is not present at all while the agents forecast using the variable that is irrelevant in the MSV, and then initialize the agents' beliefs in a small neighborhood of the RPE-consistent values.

We restrict the agents to use only one endogenous variable in their forecasting models - either a lag of inflation or a lag of output gap. Under the assumption of *i.i.d.* shocks the agents then choose between two models, based on their forecasting performance:

$$\pi_t = \alpha_\pi^\pi + \beta_\pi^\pi \pi_{t-1}, \tag{33}$$

$$\pi_t = \alpha_\pi^y + \beta_\pi^y y_{t-1}, \tag{34}$$

with the coefficients determined by the respective regressions. We call the mis-specified rule in (33) M_π, and the MSV-consistent one in (34) is denoted M_y.[8] The equilibria induced by these forecasting rules are also called M_π and M_y.

The ALM when the agents are using the M_π forecasting rule is given by the following equations, with the coefficients defined in Appendix E in equations (E12)-(E19):

$$\pi_t = \bar{a}_\pi + \bar{b}_\pi \pi_{t-1} + \bar{c}_\pi y_{t-1} + \bar{\eta}_\pi^g g_t + \bar{\eta}_\pi^u u_t, \tag{35}$$

$$y_t = \bar{a}_y + \bar{b}_y \pi_{t-1} + \bar{c}_y y_{t-1} + \bar{\eta}_y^g g_t + \bar{\eta}_y^u u_t. \tag{36}$$

We measure the forecasting performance of rules M_π and M_y by the mean squared forecast errors (MSFE). We define the error term for MSFE criterion as $E(\pi_t - \hat{\pi}_t)^2$, where π_t is given by the above ALM under the condition that *all* agents were using (33) to form the expectations.

Proposition 5 summarizes the conditions for M_π - a mis-specified rule, to be Restricted Perceptions Equilibrium.

Proposition 5. *For the model described in (26-27), M_π equilibrium in (33) 1) exists; 2) is weakly E-stable under condition (E39); 3) M_π produces MSFE that is smaller than M_y MSFE when condition (E44) holds.*

Proof. Appendix E. □

5. Dynamics of Sparse AL

In this section we study the convergence of agents beliefs to the MSV consistent rule. In particular, we are interested in the question: will the agents using penalized RLS in (8) eventually learn that the coefficient on π is zero, and converge to the MSV-consistent equilibrium? If they do, then global strong E-stability of MSV REE is supported.

The derivation of attention weights resembles that for learning about the Fisher equation, that is why we leave the weights derivation to the Appendix F.

As shown in the in Appendix D when the agents do not face attention costs the asymptotic strong E-stability for the MSV REE is obtained when the condition (D16) is satisfied. The numerical analysis in the next section supports the likely global convergence for zero attention costs.[9] We further study the global strong E-stability with positive attention costs.

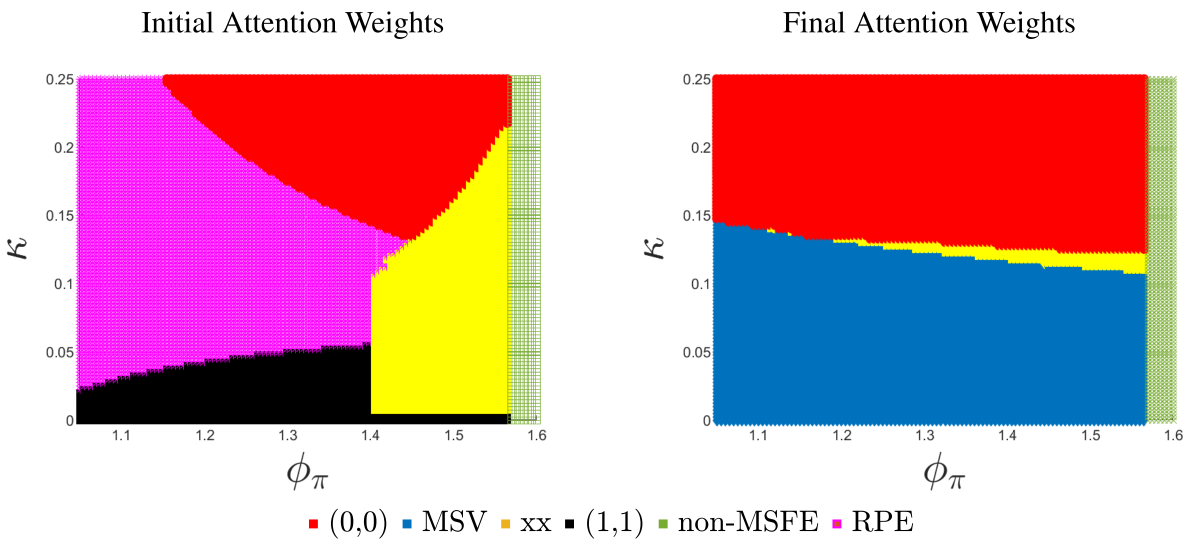
5.1 Convergence to the MSV REE

Figure 2 presents the start and end points of the Sparse AL learning dynamics, equations (8), for different values of the attention cost κ and the policy rule's aggressiveness ϕ_π. The left panel presents initial weights, obtained by a single application of the penalized regression (3) in the RPE M_π. When the derived attention costs are zero, or close to zero, the agents initially choose $(m_\pi, m_y) = (1,1)$ to

[8] We call M_y an MSV-consistent rule because it uses the same lagged endogenous variable - output gap -as the MSV REE. Therefore, in what follows we use 'MSV-consistent' and 'MSV REE' interchangeably.

[9] Outside of the MSV REE, the approximating Strong E-Stability ODE is nonlinear; therefore we cannot prove analytically that the convergence is global.

Figure 2: Attention Weights



pay full attention to both variables - black squares in the figures. There is a wide region of the parameter space at the lower values of ϕ_π where the initial weights are consistent with the RPE, $(m_\pi, m_y) = (1, 0)$, pink asterisks. Here, the restricted choice of the variables in the forecasting rule - only π - is reconfirmed by sparse rationality. This region is located at intermediate values of the attention costs and low to medium monetary policy aggressiveness. For low values of attention cost as well as for intermediate κ combined with higher ϕ_π, both the inflation and output gap tend to have non-zero (yellow area) weights. Also, for even higher attention costs, especially when the Taylor rule is very aggressive, the agents tend to disregard both π and y and forecast inflation using an anchored rule, red area. Finally, the highest values of the aggressiveness by the central bank lead to the RPE M_π forecasting rule producing worse forecasts than the alternative M_y one; thus RPE does not exist in the area of green squares. Notably, the MSV-consistent equilibrium is never an optimal solution. At best, the agents give some weight to the 'correct' variable (output gap), but they never replace the 'wrong' (inflation) with the 'correct' variable. However, this is only a static outcome.

In this paper we initialize the Sparse AL learning algorithm in two ways. In both ways for every point on the (κ, ϕ_π) grid, the agents start with the attention weights $\boldsymbol{m}$ obtained from the single run of optimization problem in (3) when the data is generated by the RPE ALM (35)-(36).[10] The vector of belief coefficients $\boldsymbol{\beta}$ is either taken from the same regression, and thus is consistent with the RPE ALM, or it is equal to that at the RPE PLM. This procedure results in two sets of combined initial beliefs $(\beta_\pi, \beta_y, m_\pi, m_y)$ which are very far away from those one would observe at the MSV REE for these values of (κ, ϕ_π). Importantly, given our initialization procedure, the agents' initial PLM would always include lagged π, the variable that is not present in the MSV REE.

We then trace the approximating ODE for a sufficiently long time[11] and observe the final beliefs and weights. The final weights are significantly more uniform than the initial ones. The weights converge to the MSV REE (blue area) for a wide range of parameters, to the anchored forecasting rule (red area), or are on the way towards theanchored rule (yellow area) with both variables having

[10] If one or both weights are equal to zero, for technical reasons we initiate them with a small positive number ε, as otherwise computation of the matrix Σ in (E33) becomes impossible.
[11] We typically use T=30, which in the case of a small constant gain $g = 0.01$ is equivalent to 3,000 periods. In the case of RLS with gains $g_n = 1/n$, continuous time of 30 is equivalent to 1e+13 periods.

non-zero weights. The RPE M_{π} does not survive for any combination of the parameters (ϕ_{π}, κ). Naturally, convergence to the anchored rule is observed for high values of κ, while the MSV REE is the limit point for lower values of the attention costs. Convergence to the MSV is observed even for many initial points with zero output gap weight. Thus, the agents who are allowed to continue learning, while taking into account the attention penalty, are still able to learn the true equilibrium, unless the attention cost is too high.

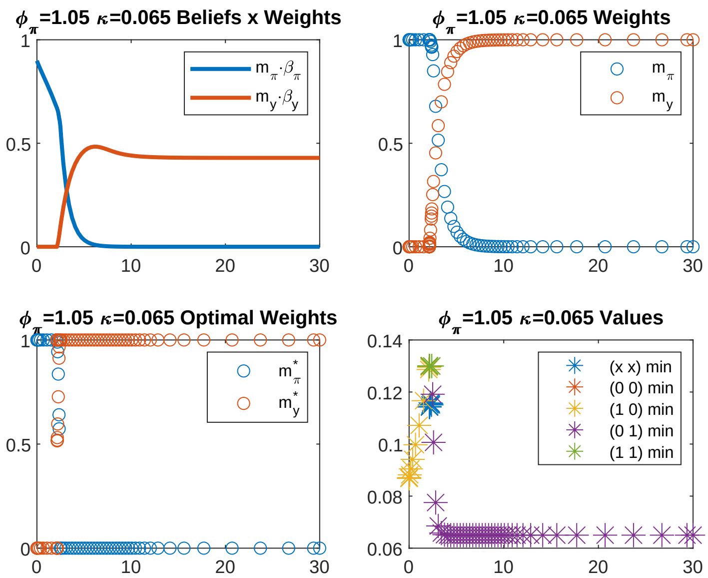
Our results do not depend on whether the initial beliefs are consistent with the RPE ALM or RPE PLM: the eventual outcome of learning is the same for both ways of initialization. This result suggests that not only the MSV REE could be *strongly* E-stable under the attention cost constraint, as we start from over-parametrized PLM that always includes positive coefficient on a variable that is not present at the MSV, but also that it could be *globally strongly* E-stable in a large region of the parameter space, with the initial beliefs very far from the MSV ones. In particular, with the second initialization method the agents start by having almost zero beliefs in the correct MSV variable y and positive beliefs in the incorrect variable π that is absent in the MSV solution, and still converge to it. However, as mentioned previously, we cannot prove the global stability analytically or numerically due to nonlinearity and high dimensionality of the problem.

Finally, we note that the conduct of monetary policy affects the asymptotic equilibrium to which our learning algorithm converges. With stricter monetary policy, agents switch to the anchored only rule (yellow and red area in the Figure 2) for smaller κ. The reason for this is that more aggressive reaction of the central bank to inflation lowers volatility and hampers predictability of both the inflation and the output gap, which reduces the benefits of paying attention to these variables while the attention costs are the same. Therefore, using the correct variable (output gap) for predicting inflation is justified only for lower values of the cost.

The exact evolution of the PLM beliefs and the attention weights leading to the switch from the RPE, $(m_{\pi}, m_y) = (1,0)$, to the MSV REE, $(m_{\pi}, m_y) = (0,1)$, is also of interest. Figure 3 presents the evolution of the PLM beliefs, optimal and actual attention weights, as well as the product of the PLM coefficient and the attention which determines the impact of a specific variable on the agents' inflation forecast. The agents start from the RPE PLM, so that their attention weights are equal to (1,0), see the starting point in the upper right panel. The RPE is a corner solution. When we allow *both* beliefs to be adjusted, β_{π} starts to decline while β_y increases, see the top panels. However, as (1,0) remains the optimal solution to (3) - see the lower left panel, - the agents continue to predict inflation using only the inflation lag. Around $t = 2.5$ the output gap belief becomes so large that it now is optimal to start paying some attention to the output gap variable. The lower right panel shows the optimal selected model - $V(m_{\pi}, m_y)$: a combination of weights that minimize the objective function in (3). The optimal solution switches from (1,0) (yellow asterisks) to (0,1) (purple asterisks) after a short transition interval where having both attention weights positive is optimal (blue, red, and green asterisks).

This evolution of the agents' PLM allows us to comment on the relationship between E-stability of the RPE, established by Audzei and Slobodyan (2022), and the strong E-stability of the MSV described above. In the RPE, the agents include *only* the inflation variable into their forecasting rule. The approximating ODE that allowed us to establish E-stability of the RPE is thus 1-dimensional in the beliefs ($\boldsymbol{\beta}$) space. The implicit attention weights were fixed at (1,0). In contrast, once the agents start learning subject to attention costs, they explicitly take into account that there could be two variables in their PLM, and thus the approximating ODE becomes 2-dimensional in $\boldsymbol{\beta}$ space. In addition, there are dynamics in the $\boldsymbol{m}$ space which were not present in the analysis of RPE E-stability. In other words, the nature of the dynamic adjustment of beliefs (and of attention weights) changes dramatically, in particular through expansion of dimensionality. In the Figure 3 the agents first utilized the second dimension of the PLM beliefs, by decreasing β_π and increasing β_y, and then moved away from (1,0) and towards (0,1) in the attention weights space. Thus, even if the initial belief on output gap in the agents' PLM is zero, they still *could* move along that dimension, while during convergence to the RPE implicit in the RPE E-stability derivation, the β_y dimension didn't exist. An even simpler process of adjustment towards the MSV REE is presented in the

Figure 3: Convergence of Weights and Beliefs



Note: The figure illustrates the convergence of weights to MSV REE consistent values for attention costs $\kappa = 0.065$ and monetary policy reaction to inflation $\phi_\pi = 1.05$. Continuous time units of the approximating ODE are on the horizontal axis. Asymptotic convergence to MSV REE is observed, with the MSV becoming the optimal solution after $t = 2.5$.

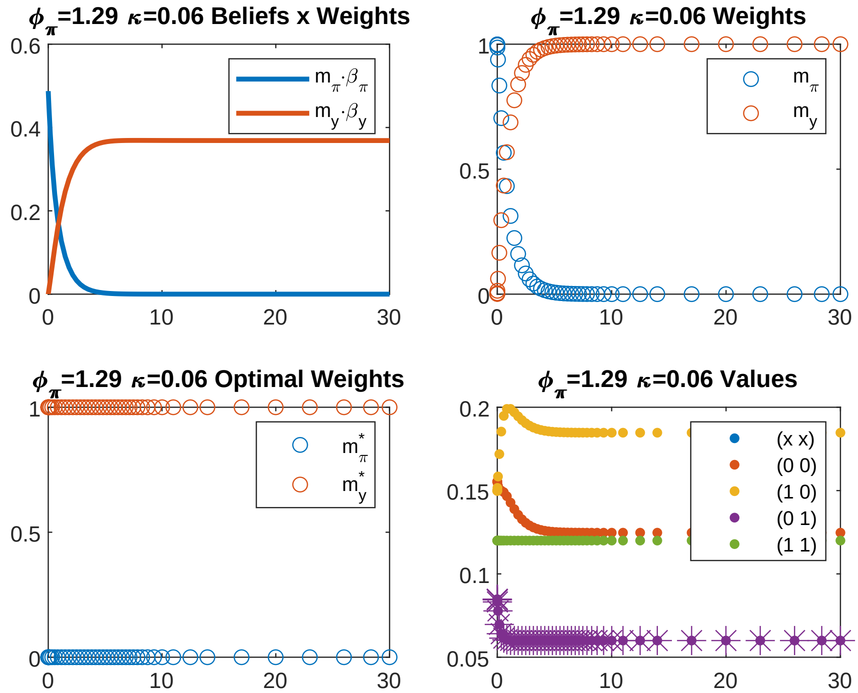
Figure 4. Along this trajectory the optimal attention weights remain equal to (0,1) from the very beginning, and the actual attention weights are adjusted monotonically to their limit values. The weight on inflation m_π declines while the weight on output gap m_y is monotonically increasing. Asymptotically the total impact of the output gap on the agents' inflation forecast, $m_y \cdot \beta_y$, upper left panel, is the same as the corresponding value at the MSV REE despite the attention costs. The agents pay the costs but still prefer to use the correct forecasting rule and the correct coefficient in it. The lower right panel shows that the corner solution $(0,1)$ remains optimal (violet asterisks) throughout the whole trajectory.

We now turn to discussion of the thin yellow wedge on the final weights panel of Figure 2, which exhibits rather non-trivial dynamics with switching of the forecasting rule's functional form.

5.2 Sliding Dynamics

For most trajectories that converge to the $(0,0)$ limit weights we observe monotonic convergence similar to that in (4): the corner solution $(0,0)$ remains optimal for the whole duration of the simulation. However, there are other types of trajectories that exhibit a switching behavior of the optimal solution in the attention weights space, and we now turn to the detailed discussion of these solutions. These are the trajectories converging to the yellow marks, see Figure 5 for an example. At some

Figure 4: Convergence of Weights and Beliefs



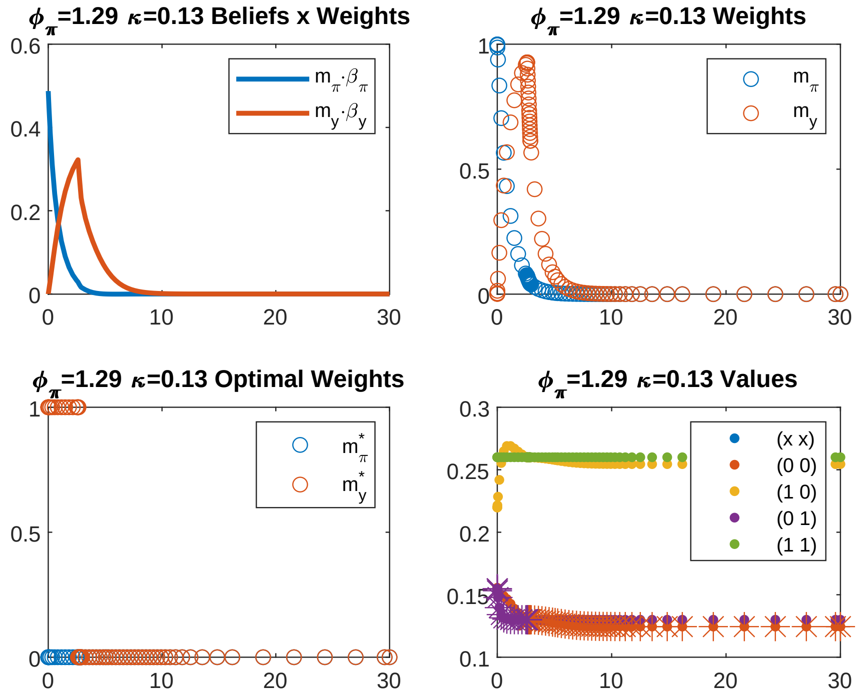
Note: The figure illustrates the convergence of weights to MSV REE consistent values for attention costs $\kappa = 0.06$ and monetary policy reaction to inflation $\phi_\pi = 1.29$. Continuous time units of the approximating ODE are on the horizontal axis. Monotonic convergence to MSV REE is observed. The bottom right graph shows the value of value functions (which exist) with the asterisks denoting the optimal (minimal) value function.

point along the trajectory around t=2.5, the value of the objective function in (3) obtained for the MSV-consistent weights, V(0,1), becomes equal to the value generated by anchored rule weights, V(0,0). After this point, the monotonic convergence to the MSV weights is replaced with a convergence to the anchored rule $(0,0)$. The switch is best seen in the lower right panel of the Figure 5: Before $t \approx 2.5$ it is the (0,1) solution that produces the minimal value (violet asterisks), but after this time it's the (0,0) solution which becomes the best (orange asterisks).

The points where such a situation happens form a surface in the $(\boldsymbol{\beta}, R, \boldsymbol{m})$ space. At one side of the surface, we have V(0,0)>V(0,1), while at the other side the opposite situation takes place. The optimal weights, respectively, are (0,1) and (0,0). From the ODE for attention weights (8) we then see that the right-hand side in the equation for m_y is discontinuous at this boundary. Importantly, it could happen that due to this discontinuity the flow described by (8) points back to the boundary on *both* sides of it, making it intuitively clear that locally the ODE trajectories will be attracted to the boundary.

In order to study the dynamics in this case, we need to turn to the theory of non-smooth differential equations described in Appendix G. As is described there, *sliding* dynamics could occur along the boundary on which the two solutions to the problem (3) give exactly the same value. This happens when the flow described by (8) points in the direction of the boundary on both sides of it, making possible a stable trajectory that lies entirely within the boundary for some time interval.

Figure 5: Convergence of Weights and Beliefs



Note: The figure illustrates the convergence of weights to $(m_\pi, m_y) = (0,0)$ for selected values of attention costs $\kappa = 0.13$ and monetary policy reaction to inflation $\phi_\pi = 1.29$. Continuous time units of the approximating ODE are on the horizontal axis. Convergence to the anchored forecasting rule. The bottom right graph shows the value of value functions (which exist) with the asterisks denoting the optimal (minimal) value function.

In our case, we observed several types of the trajectories encountering the boundary between the solutions $(m_\pi, m_y) = (0,0)$ and $(0,1)$. Two were the most common. The first type encounters the boundary, punches through it (the scalar product of projections of the flow on the normal to the boundary from two sides is positive), and continues evolving according to the ODE (8) towards the anchored forecasting rule. These are the yellow diamonds in Figure 2. Another type encounters the boundary and settles into the sliding dynamics as the scalar product of projections on the normal is negative, eventually converging to the anchored forecasting rule. All points represented by the yellow circles in Figure 2 denote such dynamics. Occasionally, a trajectory that first punched through the boundary, then encountered it for a second time and settled for the sliding dynamics, was observed. We also encountered a few trajectories whereby the sliding dynamics ended before time $T = 30$ and the trajectory then continued along the non-boundary ODE (8). Up to three episodes of sliding could occur along the convergence trajectory for some parameter values.

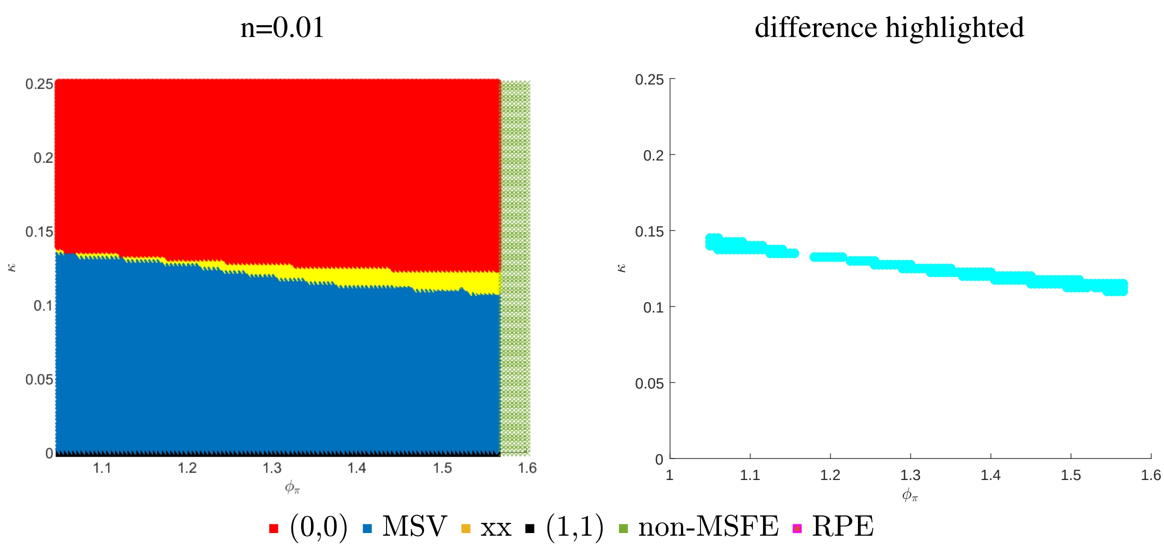
Importantly, no trajectories that encountered the boundary were observed to converge to the MSV REE attention weights, whether or not the trajectory was converging to the MSV before the encounter.

5.3 The Slow Weights Learning Case and Non-Smooth Dynamics

The case $n = 1$ assumes that the agents update attention weights with the same speed as their OLS beliefs. One, however, could entertain different hypotheses. Generally speaking, reconsideration of the set of variables to be included into the forecasting rule and of the attention weights for different variables is a significantly more complex task than updating R, computing its inverse and multiplying it by the forecast error to get the iteration of $\boldsymbol{\beta}$. Determining optimal weights is a constrained optimization problem that in a multi-dimensional case requires comparison of multiple corner solutions. Therefore, it is reasonable for the agents to reconsider their weights less frequently than their OLS beliefs. This would then amount to $n \ll 1$ in the updating equations (7-8).

We present the results of the relatively slow learning of attention weights ($n = 0.01$) in Figure 6. The left panel shows the final weights, while on the right we show the difference with the results for $n = 1$. Blue points are the parameter values for which in the case $n = 1$ we had convergence to the MSV REE, but with $n = 0.01$ the anchored forecasting rule is the final outcome. All of these trajectories that have switched the final attention weights are those that encounter the boundary between (0,1) and (0,0) solutions along the way. In order to understand this behavior we look carefully into a simplified version of the model dynamics.

Figure 6: Attention Weights



In order to generate the intuition for the results on the slow sliding dynamics we inspect the equations (8) and see that what matters for the dynamics of both beliefs and attention weights are the element-wise products of attention weight and belief vectors $\boldsymbol{m} \odot \boldsymbol{\beta}$, which are the total effects of the two variables $(m_x \cdot \beta_x)$ on the overall inflation forecast. For the trajectories where the optimal attention weight on inflation, m_π, remains equal to 0, only the impact $\Psi_y = m_y \cdot \beta_y$ matters. Therefore, in order to simplify the exposition we switch our attention to the dynamics in a two-dimensional space $\underline{x} = (m_y, \beta_y)$.[12] The ODE (8) in this space is given by the following equations:

$$\begin{aligned} \dot{\beta_y} &= \bar{c}_\pi - m_y \cdot \beta_y, \\ \dot{m_y} &= n \cdot \left(m_y^* - m_y\right). \end{aligned} \tag{37}$$

In this space, the boundary between the two corner solutions (0,0) and (0,1) is given as the solution to the equation $V(0,0) = V(0,1)$ which is $\Psi_y = m_y \cdot \beta_y = \bar{\Psi}_y$, a hyperbola in the two-dimensional

[12] We further assume that the second moments R have converged to their equilibrium values Σ in order to simplify the exposition.

space (m_y, β_y). Using notation from Appendix G, the equation for the boundary is

$$\sigma(m_y, \beta_y) = m_y \cdot \beta_y - \bar{\Psi}_y = 0. \tag{38}$$

The discontinuity above and below the boundary comes from the fact that the optimal solution for m_y is either 0 or 1 at different sides of it. For $\sigma(m_y, \beta_y)$ below the boundary, $m_y^* = 1$ while above the boundary $m_y^* = 0$.[13] We write the time derivative of σ as

$$\dot{\sigma} = \left(m_y \dot{\cdot} \beta_y\right) = \dot{m_y} \cdot \beta_y + m_y \cdot \dot{\beta_y} \tag{39}$$
$$= \left(\bar{c}_\pi - \bar{\Psi}_y\right) \cdot m_y + n \cdot \left(m_y^* - m_y\right). \tag{40}$$

The first term in the last line is always positive, while the second is negative above the boundary, where $m_y^* = 0$, and positive below it, because $m_y^* = 1$. When the second term is larger in absolute value than the first, the boundary is stable, as $\dot{\sigma}$ is negative for $\sigma > 0$ and positive for $\sigma < 0$. Sliding dynamics ensue. However, when we decrease n, the second term becomes smaller in the absolute value. It is now possible to have $\dot{\sigma} > 0$ also for $\sigma > 0$, and there is no sliding as the boundary is simply punched through.

With sliding, the system evolves along the boundary $\sigma(\underline{x}) = 0$. Given that the time derivative of β_y is a positive constant at the boundary, the value of β_y grows without bounds during sliding. However, as the product of β_y and m_y at the boundary is constant, m_y must converge to zero. Therefore, the limit point of the sliding dynamics in this simple case could only be $(m_\pi, m_y) = (0, 0)$. This behavior is probably responsible for the fact that once the sliding dynamics commences in our simulations that take place in 7D space, the anchored solution (0,0) is the ultimate outcome, even when the sliding is consequently discontinued: sliding brings the trajectory ever closer to (0,0) rather than back to the (0,1) solution, the MSV REE.

Another consequence of the slow updating of attention weights consists of affecting whether the trajectory even reaches the (0,0)-(0,1) boundary. Outside of the simplified 2D case we just considered, the boundary is a complicated object in the seven-dimensional space rather than a simple hyperbola $\Psi_y = m_y \cdot \beta_y = \bar{\Psi}_y$. It is possible that when the trajectory is moving towards $m_y = 1$ very fast ($n = 1$) hitting the boundary becomes impossible, thus expanding the region in the parameter space where convergence to the MSV REE is observed.

6. Conclusions

In this paper, we extend the standard Recursive Least Square learning algorithm to the case of penalized regression as in Gabaix (2014). We investigate the convergence properties of the continuous time approximating ODE for this combined algorithm, called Sparse Adaptive Learning, and establish that allowing for dynamic choices of attention to be paid to different model variables rules out convergence to the RPE. The attention weights corresponding to the RPE are never the ultimate outcome, even though initially the beliefs are consistent with the RPE. This result is in stark contrast with a single application of the sparsity penalized regression, which never delivered the MSV REE as an outcome in Audzei and Slobodyan (2022). The global E-Stability of MSV REE we demonstrate in the paper implies that even when the agents, who are allowed to reconsider their forecasting

[13] This is because the value of the penalty term is increasing in m_y, and so the forecasting rule with fewer variables is preferred when we increase m_y marginally from the boundary.

rule choices in a self-referential system subject to attention costs, initiate learning from the 'wrong' equilibrium, they still typically learn the MSV REE. Alternatively, in a system with little volatility or autocorrelation of the endogenous variables, they will switch to using the anchored rule. This result also raises doubts regarding the RPE as the outcome of some learning process, because its existence relies on the agents using only the variables present in the restricted information set and ignoring others along the transition trajectory. Whether this fragility of the RPE is a general result in a wider set of self-referential models remains an interesting topic for future research.

The learning algorithm considered in this paper could lead to non-smooth dynamics due to the agents discontinuously selecting the set of variables to be included into their forecasting rule. During such discontinuous steps, the number of variables in the agents' information set changes. The presence of these non-smooth dynamics forces us to rely on the theory of non-smooth differential equations to study the approximating ODE. We demonstrate that the presence of discontinuous jumps in the number of variables present in the forecasting rule could result in sliding dynamics, which has not been observed previously in the adaptive learning literature.

We also establish that the relative speed at which belief coefficients and attention weights are adjusted has important implications for the trajectories that could encounter the boundary between the two corner solutions, and develop analytical results in a specific restricted case. Less frequent updating of the attention weights relative to the beliefs results in marginal expansion of the region in the parameter space where the anchored forecasting rule is the asymptotic outcome of the learning.

The strictness of the monetary policy affects the evolution of the learning algorithm. When the Taylor rule is more aggressive, the convergence from the mis-specified rule to either the MSV or the anchored rule is faster. With the stricter monetary policy, the model variables become less volatile and less correlated across time, making lags of endogenous variables less useful for forecasting. In the presence of attention costs, this could lead to the agents selecting an anchored forecasting rule – anchored to the long-term inflation or inflation target – rather than the rule that is consistent with the MSV REE.

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Appendix A: Simple Model

A.1 Rational Expectations MSV

The REE MSV - consistent forecasting rule for the system (12) takes the form:

$$\pi_t = \gamma r_t \quad \Rightarrow E_t \pi_{t+1} = \gamma r_t, \tag{A1}$$

where γ denotes agents' beliefs. This results in the ALM:

$$\pi_t = \frac{1+\rho\gamma}{\phi} r_t. \tag{A2}$$

In the MSV REE, the PLM and ALM coincide:

$$\gamma = \frac{1+\rho\gamma^*}{\phi} = T(\gamma^*) \Rightarrow \gamma^* = \frac{1}{\phi-\rho}. \tag{A3}$$

Note that for $\phi - \rho < 1$, $\gamma^* > 1$.

E-Stability For the simple model, the conditions could be derived as follows.

The MSV solution is *weakly* E-Stable when $\frac{\partial T(\gamma)}{\partial \gamma} = \frac{\rho}{\phi} < 1$. This condition is satisfied for any $0 < \rho < 1$ and $\phi > 1$.

To derive the condition for *strong* E-stability, we assume that the over-parametrized rule takes the form:

$$\pi_t = \gamma r_t + \omega \pi_{t-1}. \tag{A4}$$

With the resulting ALM being

$$\pi_t = \frac{1+(\rho+\omega)\gamma}{\phi} r_t + \frac{\omega^2}{\phi}, \tag{A5}$$

and the equilibrium conditions given by

$$\begin{aligned} \gamma^* &= \frac{1}{\phi-\rho-\omega^*}, \\ \omega^* &= \frac{(\omega^*)^2}{\phi}. \end{aligned}$$

There are two solutions for ω^*: 0 (corresponding to the MSV REE) and ϕ. The Jacobian takes the following form:

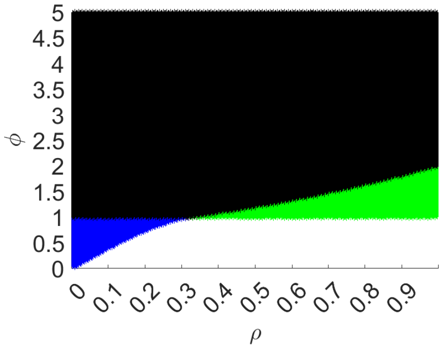
$$J = \begin{bmatrix} \frac{\rho+\omega^*}{\phi} - 1 & \frac{\gamma^*}{\phi} \\ 0 & 2\frac{\omega^*}{\phi} - 1 \end{bmatrix},$$

with the eigenvalues $(\frac{\rho+\omega^*}{\phi} - 1, 2\frac{\omega^*}{\phi} - 1)$. For $\omega^* = \phi$ both eigenvalues are positive and thus this solution is always E-unstable. For $\omega^* = 0$ one eigenvalue equals -1 <0, while the other produces the following strong E-stability condition

$$\frac{\rho}{\phi} < 1, \tag{A6}$$

which coincides with the weak E-stability condition.

Figure A1: Uniqueness of RPE



Note: In the green area E-stability and MSFE criteria are satisfied and the solution is unique. In the black area the solution is unique, but MSFE criterion is not satisfied. In the blue area there are multiple solutions and at least one of the criteria is not satisfied.

A.2 Proof of Proposition 1

A.2.1 Existence of Stable β

The agents' perceived PLM coefficient β at the RPE must be equal to the OLS coefficient. Using that $cov(\pi_t,\pi_{t-1}) = \frac{(\rho+\bar{b})\bar{c}^2\sigma_r^2}{1-\rho\bar{b}}$ and $var(\pi) = \frac{1+\rho\bar{b}}{(1-\rho\bar{b})(1-\bar{b}^2)}\bar{c}^2\sigma_r^2$, β must be the solution to the following equation:

$$\beta = \frac{cov(\pi_t,\pi_{t-1})}{var(\pi)} = \frac{\frac{(\rho+\bar{b})\bar{c}^2\sigma_r^2}{(1-\rho\bar{b})(1-\bar{b}^2)}}{\frac{1+\rho\bar{b}}{(1-\rho\bar{b})(1-\bar{b}^2)}\bar{c}^2\sigma_r^2} = \frac{\rho+\bar{b}}{1+\rho\bar{b}} = \frac{\beta^2+\rho\phi}{\rho\beta^2+\phi} = \Gamma(\beta). \tag{A7}$$

In Figure A1 for a wide parameter region, we show the area where β is unique and where the E-Stability and MSFE conditions are satisfied in green. In the black area, the solution is unique and E-Stable, but the MSFE criterion is not satisfied. In the blue area, there are multiple real solutions and neither is E-Stable. Below we formally present the conditions. From Eq. (A7) one can immediately observe that as long as $\phi > 1$

$$\begin{aligned}
\Gamma(0) &= \rho, \\
\rho < \Gamma(1) &= \frac{1+\rho\phi}{\rho+\phi} = 1 - \frac{(\phi-1)(1-\rho)}{\phi+\rho} < 1, \\
\Gamma'(\beta) &= \frac{2\beta\phi(1-\rho^2)}{(\rho\beta^2+\phi)^2} = \frac{2\phi\beta(1-\rho\beta)^2}{\phi(1-\rho^2)} > 0.
\end{aligned}$$

Thus, there must exist at least one intersection of the functions β and $\Gamma(\beta)$ in the $[0;1]$ interval. To check for more than one intersection, we re-write (A7) further as

$$\rho\beta^3 - \beta^2 + \phi\beta - \phi\rho = f(\beta) = 0, \tag{A8}$$

where we observe that

$$\begin{aligned}
f(0) &< 0, \\
f(1) &= (\phi-1)(1-\rho) > 0, \\
f(\rho) &= \rho^2(\rho^2-1) < 0.
\end{aligned}$$

As long as the Taylor principle $\phi > 1$ is satisfied, there exists at least one root of the cubic equation (A8) in the $[\rho;1]$ interval. Using Descartes' rule of signs one can easily establish that there could be either 1 or 3 positive real roots, as there are 3 sign changes in the coefficients of the $f(\beta)$ polynomial, and 0 negative real roots, as there are no sign changes of the $f(-\beta)$ polynomial. We could further compute the turning points of f as points where $f'(\beta) = 0$:

$$\beta_{1,2} = \frac{1 \pm \sqrt{1-3\rho\phi}}{3\rho}.$$

If we have $3\rho\phi > 1$, there are no turning points of f and thus no other real roots.[14]

A.2.2 RPE E-Stability

In order to prove weak E-stability of the RPE, we need to show that for β^* satisfying Eq. (A8) the value of $\frac{2\phi\beta(1-\rho\beta)^2}{\phi(1-\rho^2}$ is less than unity, which cannot be done analytically. Numerically, for all points in the open set $\{(\rho,\phi): \quad 1 > \rho > 0, \quad 5 > \phi > 1\}$ this condition is satisfied as shown as the black and green area in Figure A1.

Thus, for $\phi > 1$ the RPE is weakly E-Stable.

A.2.3 MSFE Criterion

To establish the region in the (ρ,ϕ) space where the M_π forecasting rule predicts better than the M_r rule while the M_π ALM is operative, we first derive the condition (18). Denote $\varepsilon_t^i \equiv \hat{\pi}_t^i - \pi_t$ an error of inflation forecast under a model i provided that the ALM is generated under model M_π.

$$\begin{aligned}
\varepsilon_t^\pi &= (\beta - \bar{b})\pi_{t-1} - \bar{c}r_t = (\bar{b} + \bar{c}\frac{\sigma_{\pi_{t-1},r_t}}{\sigma_\pi^2} - \bar{b})\pi_{t-1} - \bar{c}r_t, \\
E_t(\varepsilon_t^\pi)^2 &= \bar{c}^2\sigma_r^2\left(1 - \frac{\sigma_{\pi_{t-1},r_t}^2}{\sigma_\pi^2\sigma_r^2}\right), && \text{(A9)} \\
\varepsilon_t^r &= -\bar{b}\pi_{t-1} + (\hat{\gamma} - \bar{c})r_t = -\bar{b}\pi_{t-1} + (\bar{c} + \bar{b}\frac{\sigma_{\pi_{t-1},r_t}}{\sigma_r^2} - \bar{c})r_t, \\
E_t(\varepsilon_t^r)^2 &= \bar{b}^2\sigma_\pi^2\left(1 - \frac{\sigma_{\pi_{t-1},r_t}^2}{\sigma_\pi^2\sigma_r^2}\right). && \text{(A10)}
\end{aligned}$$

Model M_π produces smaller MSFE if:

$$E_t(\varepsilon_t^\pi)^2 < E_t(\varepsilon_t^r)^2 \Rightarrow \bar{c}^2\sigma_r^2 < \bar{b}^2\sigma_\pi^2, \tag{A11}$$

which is the condtion (18) in the text. The condition (18) reduces to:

$$\frac{\bar{b}^2(1+\rho\bar{b})}{(1-\rho\bar{b})(1-\bar{b}^2)} \geq 1 \Rightarrow \rho \geq \frac{\phi^2 - 2\beta^4}{\phi\beta^2}. \tag{A12}$$

As $\beta = \beta(\rho,\phi)$ the condition (A12) is a complicated object in (ρ,ϕ) space. We consider a solution for the root of β at the E-Stability boundary where $\phi = 1$. The (A8) can then be factored as $(\beta - 1)\cdot(\rho\beta^2 - (1-\rho)\beta + \rho) = 0$. There is always a unit root. In addition, for $\rho \leq 1/3$, the discriminant of the quadratic term is positive and has another root in the $[\rho;1]$ interval. For $\rho > 1/3$, the discriminant

[14] Further, one can use the Sturm's theorem to show that for all ϕ such that $3\rho\phi < 1$ and $1 < \phi < 2$ there is only one root in the $[\rho;1]$ interval; however, the derivations become cumbersome.

of the quadratic term is negative, and there are only complex roots, so the only RPE solution is $\beta = 1$. Close to the boundary $\phi = 1$ the agents continue believing that inflation is a near unit root process, $\bar{b} \approx 1$ and so the variance of inflation is very large. Lagged inflation thus explains a significantly larger share of the variance than the real interest rate, and so the M_π forecasting rule has better performance than the M_r one.

Consider now what happens to the unique root for the values $\{(\rho, \phi): \quad \phi = 1 + \varepsilon, \rho > 1/3\}$. Taking the full differential of (A8) with respect to ϕ and plugging in the values of $\phi = \beta = 1$, we obtain

$$\begin{aligned} \beta_\phi &= \frac{1-\rho}{1-3\rho} < 0, \\ \bar{b}_\phi &= 2\beta_\phi - 1 < 0. \end{aligned}$$

Therefore, as ϕ increases above unity, the value of β and $\bar{b}$ starts to drop.

We now consider the boundary where the forecasting performance of the two single equation rules is exactly the same. Setting the LHS of Eq. (A12) equal to one and solving the resulting quadratic equation, we obtain that the value of $\bar{b}$ on the boundary is decreasing in ρ. As $\bar{b}_\phi < 0$, this means that, in order to stay on the boundary as ϕ increases, ρ must increase as well. Thus, the boundary where the forecasting rules M_π and M_r perform exactly the same is upward-sloping in the (ρ, ϕ) space. Below the boundary M_π is better, as discussed above. This is the green area in Figure A1.

For $\rho = 1/3$ and $\phi = 1$ the equation (A8) has a triple root of unity. As ρ decreases below 1/3, one of the roots drops while another becomes larger than unity. Taking the lower root as the solution, with β and $\bar{b}$ decreasing, the condition (A12) is no longer satisfied for a value of $\underline{\rho} \approx 0.328$, which defines the intersection of the (A12) boundary with the horizontal line $\phi = 1$ and determines the initial point of the border between the green and the black area in Figure A1. To derive the value of $\underline{\rho}$, plug $\phi = 1$ into (A7) and (A12) evaluated at equality:

$$\rho = \frac{\beta}{1+\beta+\beta^2}, \tag{A13}$$

$$\rho = \frac{1-2\beta^4}{\beta^2}. \tag{A14}$$

While (A13) is a non-decreasing function of β with $\rho(0) = 0$ and $\rho(1) = 1/3$, (A14) describes a decreasing function of β with $\rho(0) \to +\infty$ and $\rho(1) = -1$; therefore, there is a single intersection in the $[0;1]$ interval of β. The equation for β that we then obtain by setting expressions (A13) and (A14) equal to each other is a polynomial of order 6. By plugging its unique real root from $[0;1]$ interval into (A13), we obtain a numerical solution for $\underline{\rho}$ mentioned above.

A.3 Proof of Proposition 2

Suppose the agents use two variables in their PLM: r_t and π_{t-1}, such that the PLM is an RPE-consistent with $\gamma = 0$. The 'data' that the agents see is created by the RPE ALM, and we assume that the agents are using the second-moments matrix R that is consistent with this ALM.

The updating differential equation for beliefs is given by the E-stability ODE where the functional forms of the PLM and ALM coincide:

$$\frac{d}{d\tau}\begin{bmatrix}\beta \\ \gamma\end{bmatrix} = \begin{bmatrix}\bar{b}(\beta) - \beta \\ \bar{c}(\beta, \gamma) - \gamma\end{bmatrix}. \tag{A15}$$

One can immediately see that $\bar{b}(\beta)-\beta = \frac{1}{\phi}\beta^2 - \beta = \beta\left(\frac{\beta}{\phi}-1\right) < 0$ as typically $\phi > 1 > \beta > 0$. Similarly, looking at the second line of (A15), it is clear that $\bar{c}(\beta,\gamma)-\gamma = \frac{1}{\phi} - \gamma > 0$ as long as the PLM coefficient is not too different from the RPE value of $\gamma = 0$. At the RPE, the agents start with $\gamma = 0$, which means that the derivative is positive.

In case the agents do not exactly know the matrix of second moments, the RHS of the E-stability ODE will be given by:

$$\frac{d}{d\tau}\begin{bmatrix}\beta\\ \gamma\end{bmatrix} = R^{-1}\cdot\Sigma\cdot\begin{bmatrix}\bar{b}(\beta)-\beta\\ \bar{c}(\beta,\gamma)-\gamma\end{bmatrix}. \tag{A16}$$

As long as R, the agents' beliefs about the DGP in which they live is not too far from the true DGP that is summarized in Σ, we still have a derivative of β negative and of γ positive. □

A.4 Dynamics

Solving (3) results in nine possible cases, with the corresponding value functions denoted as $V(m_\pi, m_r)$. V reflects the agents' forecast errors, which the agents minimize in (3). When a weight takes the value $\in [0;1]$ we mark it as x in the name of the value function. For example, we denote the inner solution as $V(x,x)$. The inner solution takes the form:

$$m_r = 1 - \frac{\kappa}{(\bar{c}_s)^2\left(1-R^2\right)}\frac{\bar{b}-\bar{c}_s R}{\bar{b}}, \tag{A17}$$

$$m_\pi = 1 - \frac{\kappa}{\bar{b}^2\left(1-R^2\right)}\frac{\bar{c}_s-\bar{b}R}{\bar{c}_s}, \tag{A18}$$

$$V(x,x) = 2\kappa - \frac{\kappa^2(\bar{b}_\pi^2 - 2R\bar{c}_s\bar{b} + (\bar{c}_s)^2)}{2(\bar{b}_\pi^2(\bar{c}_s)^2 - R^2\bar{b}^2(\bar{c}_s)^2)}. \tag{A19}$$

Clearly, if the weights from (A18) and (A17) are within the $[0;1]$ interval, $V(x,x)$ is the minimum. When one or both of the weights are outside $[0;1]$, we define the rest of the value functions for corner solutions in (A20)-(A27).

$$[m_\pi = 0, 0 < m_r < 1] \qquad V(0,x) = \frac{1}{2}\bar{b}^2 + \kappa - \frac{(R\bar{b}\bar{c}_s - \kappa)^2}{2(\bar{c}_s)^2}, \tag{A20}$$

$$[0 < m_\pi < 1, m_r = 0] \qquad V(x,0) = \frac{1}{2}(\bar{c}_s)^2 + \kappa - \frac{(R\bar{c}_s\bar{b} - \kappa)^2}{2\bar{b}^2}, \tag{A21}$$

$$[m_\pi = 0, m_r = 0] \qquad V(0,0) = \frac{1}{2}((\bar{c}_s)^2 + \bar{b}^2 + R\bar{b}\bar{c}_s), \tag{A22}$$

$$[m_\pi = 1, m_r = 0] \qquad V(1,0) = \frac{1}{2}(\bar{c}_s)^2 + \kappa, \tag{A23}$$

$$[m_\pi = 0, m_r = 1] \qquad V(0,1) = \frac{1}{2}\bar{b}^2 + \kappa, \tag{A24}$$

$$[m_\pi = 1, m_r = 1] \qquad V(1,1) = 2\kappa, \tag{A25}$$

$$[m_\pi = 1, 0 < m_r < 1] \qquad V(1,x) = 2\kappa - \frac{\kappa^2}{2\bar{c}_\sigma^2}, \tag{A26}$$

$$[0 < m_\pi < 1, m_r = 1] \qquad V(x,1) = 2\kappa - \frac{\kappa^2}{2\bar{b}^2}. \tag{A27}$$

Proof of Proposition 3. **Boundaries for small values of κ.** The boundary for switching from $V(1,0)$ to $V(1,x)$ b_2 is defined from the existence of $V(1,x)$ as $V(1,x) < V(1,0)$ whenever there exists weight on real interest rate $0 \leq m_r \leq 1$

$$1 < 1 - \frac{\kappa}{\bar{c}_s^2} < 0 \Rightarrow \frac{\kappa}{\bar{c}_s^2} < 1 \Rightarrow \frac{\kappa(1+\rho\bar{b})}{(1-\rho\bar{b})(1-\bar{b}^2)} < 1, \tag{A28}$$

where we have used the fact that $\kappa > 0$ and $\bar{c}_s^2 > 0$. Consider the Eq. (A28) as equality, and write the solution to it as $\bar{\kappa}_2(\bar{b})$. Obviously, $\bar{\kappa}_2(0) = 1$ and $\bar{\kappa}_2(1) = 0$. Moreover, $\frac{\partial \bar{\kappa}_2}{\partial \bar{b}} < 0$ everywhere, as the numerator of the derivative contains 3 terms that are always negative for $0 \leq \bar{b} \leq 1$ and the denominator is positive. Therefore, the function $\bar{\kappa}_2(\bar{b})$ is monotonically decreasing. This means that the function $\bar{\kappa}_2(\bar{b})$ has a unique monotonically declining inverse $\bar{b}_2(\kappa)$ such that $\bar{b}_2(0) = 1$ and $\bar{b}_2(1) = 0$.

The threshold b_1 is defined as $m_\pi\beta$ such that $V(0,1) < V(x,1)$, when adding a positive weight on lag of inflation is no longer optimal. Again, as $V(x,1) < V(0,1)$ whenever there exists $0 \leq m_\pi \leq 1$, the threshold is given by the condition $0 \geq m_\pi$ or $m_\pi \geq 1$:

$$1 - \frac{\kappa}{\bar{b}^2} \leq 0 \Rightarrow \bar{b}^2 \leq \kappa \Rightarrow m_\pi\beta \leq \sqrt[4]{\phi^2\kappa}, \tag{A29}$$

where in the derivations we have used the fact that $\kappa > 0$ and $\phi > 0$. Thus, $b_1 = \sqrt[4]{\phi^2\kappa}$.

The condition for the boundary b_1 (A29) is given as a monotonically increasing function $\bar{\kappa}_1(\bar{b}) = \bar{b}^2$, or alternatively its unique inverse $\bar{b}_1(\kappa) = \sqrt{\kappa}$. This function is monotonically increasing, starts at the origin and reaches unity at $\kappa = 1$. Therefore, there is a single intersection at some $1 > \bar{\kappa}^* > 0$. For $\kappa \leq \bar{\kappa}^*$ we then immediately have that $\bar{b}_1 \leq \bar{b}_2$.

Moreover, the condition $\bar{\kappa}_2(\bar{b}) = \bar{\kappa}_1(\bar{b})$ yields the quadratic equation $2\bar{b}^2 + \rho\bar{b} - 1 = 0$ that has a unique solution in $[0;1]$ interval for any $\rho \in [0;1]$. This solution is related to the horizontal coordinate of the peak of the green area, where the boundaries of the $(0,1)$ and $(1,0)$ regions merge.

Boundaries for large values of κ. Boundary b_3 is defined as a threshold between the areas where the optimal solution is either a constant only $(0,0)$ or MSV-consistent $(0,1)$:

$$\frac{1}{2}(\bar{c}_s^2 + \bar{b}_\pi^2 + R\bar{b}\bar{c}_s) < \frac{1}{2}\bar{b}^2 + \kappa \Rightarrow \kappa > \frac{1}{2}\frac{(1-\bar{b}^2)}{(1+\rho\bar{b})}(1-\rho\bar{b}+\bar{b}). \tag{A30}$$

□

Appendix B: New Keynesian Model Derivations

Households maximize the infinite discounted sum of utility over consumption C_t and labour decisions N_t:

$$\sum_{t=0}^{\infty} \beta^t U(C_t, H_t, N_t), \tag{B1}$$

where $H_t = hC_{t-1}$ is external habit and $0 < \beta < 1$ is the discount factor. The optimization results in the familiar conditions:

$$-\frac{U_{n,t}}{U_{c,t}} = \frac{W_t}{P_t}, \tag{B2}$$

$$Q_t = E_t\left[\frac{U_{c,t+1}}{U_{c,t}}\frac{P_t}{P_{t+1}}\right]. \tag{B3}$$

The first equation equalizes agent's utility of consumption and dis-utility of labour. The second is an Euler equation determining agents' inter-temporal consumption decisions. We assume utility separable in consumption and labour, with σ - relative utility of risk aversion, ϕ - Frisch elasticity of labour supply, and g_t a preference shock:

$$U_t = e^{g_t}\left(\frac{(C_t - H_t)^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\phi}}{1+\phi}\right) \tag{B4}$$

so that

$$U_{c,t} = e^{g_t}(C_t - H_t)^{-\sigma}. \tag{B5}$$

Plugging $Y_t = C_t$ into the linearized Euler equation, we obtain the investment-savings curve (23).

For the rest of the model, we utilize a textbook model from Galí (2015). The firms use labour to produce differentiated final goods and face nominal rigidities á la Calvo with the probability of optimizing a price θ. The differentiated good is aggregated using a consumption index: $C_t = \left[\int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}$. Firms' pricing decisions result in a new Keynesian Phillips curve in (24) with $\omega \equiv (1-\theta)(1-\beta\theta)(\sigma+\phi)/\theta$.

It is convenient to rewrite the system of equation (26)-(27) as:

$$\begin{bmatrix}\pi_t\\ y_t\end{bmatrix} = AE_t\begin{bmatrix}\pi_{t+1}\\ y_{t+1}\end{bmatrix} + C\begin{bmatrix}\pi_{t-1}\\ y_{t-1}\end{bmatrix} + B\begin{bmatrix}u_t\\ g_t\end{bmatrix}, \tag{B6}$$

with $A = \begin{bmatrix}\beta - \frac{\omega(1-h)(\phi_\pi-1)}{(1+h)\sigma} & \frac{\omega}{1+h}\\ -\frac{1-h}{(1+h)\sigma}(\phi_\pi - 1) & \frac{1}{1+h}\end{bmatrix}$, $C = \begin{bmatrix}0 & \frac{\omega h}{(1+h)}\\ 0 & \frac{h}{(1+h)}\end{bmatrix}$, $B = \begin{bmatrix}1 & \frac{\omega(1-h)}{(1+h)\sigma}\\ 0 & \frac{1-h}{(1+h)\sigma}\end{bmatrix}$.

Appendix C: Expert Forecast of the Output Gap

We first derive output gap expectations, given the inflation PLM, which can be potentially inconsistent with REE MSV solution. Suppose inflation PLM has the following form

$$\pi_t = \psi_\pi \pi_{t-1} + \psi_y y_{t-1}, \tag{C1}$$

$$E_t\pi_{t+1} = \psi_\pi\pi_t + \psi_y y_t = \psi_\pi^2\pi_{t-1} + \psi_\pi\psi_y y_{t-1} + \psi_y y_t. \tag{C2}$$

Plugging this PLM into (26) - (27) and denoting $b_\pi^\pi = \left(\beta - \frac{\omega(1-h)(\phi_\pi-1)}{(1+h)\sigma}\right)$ and $b_\pi^y = -\frac{1-h}{(1+h)\sigma}(\phi_\pi - 1)$, we get:

$$y_t = \frac{1}{1-b_\pi^y\psi_y}\left(b_\pi^y\psi_\pi^2\pi_{t-1} + (b_\pi^y\psi_\pi\psi_y + \frac{h}{1+h})y_{t-1} + \frac{1}{1+h}E_t y_{t+1} + \frac{1-h}{(1+h)\sigma}g_t\right). \tag{C3}$$

Similarly, for inflation:

$$\begin{aligned}\pi_t = {} & \frac{b_\pi^\pi\psi_\pi^2}{1-b_\pi^y\psi_y}\pi_{t-1} + \frac{\omega+\beta\psi_y}{(1-b_\pi^y\psi_y)(1+h)}E_t y_{t+1}\\ & + \left(\frac{b_\pi^\pi\psi_\pi\psi_y}{1-b_\pi^y\psi_y} + \frac{h(\omega+\psi_y\beta)}{(1+h)(1-b_\pi^y\psi_y)}\right)y_{t-1} + \frac{(1-h)(\omega+\psi_y\beta)}{(1+h)\sigma(1-b_\pi^y\psi_y)}g_t + u_t,\end{aligned} \tag{C4}$$

where we have used $b_\pi^\pi - \omega b_\pi^y = \beta$.

We further assume that the agents receive "expert advice" which coincides with the MSV solution for output given the current ALM.[15]. Thus the expert forecasting model is:

$$y_t = \gamma_y y_{t-1} + \gamma_\pi \pi_{t-1} + \gamma_g g_t + \gamma_u u_t, \tag{C5}$$

$$y_{t+1} = \gamma_y y_t + \gamma_\pi \pi_t = \gamma_y^2 y_{t-1} + \gamma_\pi \gamma_y \pi_{t-1} + \gamma_\pi \pi_t + \gamma_y \gamma_g g_t + \gamma_y \gamma_u u_t. \tag{C6}$$

Now, because these experts know the system in (C3)-(C4), they plug the expression for inflation and rearrange using $b_\pi^\pi - \omega b_\pi^y = \beta$:

$$\begin{aligned}
&E_t y_{t+1}((1-b_\pi^y \psi_y)(1+h) - \gamma_\pi(\omega+\beta\psi_y)) \\
&= (\gamma_\pi \psi_y \psi_\pi b_\pi^\pi (1+h) + \gamma_\pi h(\omega+\psi_y\beta) + (h+1)\gamma_y^2 \left(1-b_\pi^y\psi_y\right))y_{t-1} \\
&+ \gamma_\pi(1+h)(\gamma_y(1-b_\pi^y\psi_y) + b_\pi^\pi \psi_\pi^2)\pi_{t-1} + \\
&(\gamma_y\gamma_g(1-b_\pi^y\psi_y)(1+h) + \frac{\gamma_\pi(1-h)}{\sigma}(\beta\psi_y+\omega))g_t \\
&+ (1+h)(1-b_\pi^y\psi_y)(\gamma_y\gamma_u+\gamma_\pi)u_t.
\end{aligned} \tag{C7}$$

Now, we redefine the coefficients such that

$$E_t y_{t+1} = \tilde{\gamma}_y y_{t-1} + \tilde{\gamma}_\pi \pi_{t-1} + \tilde{\gamma}_u u_t + \tilde{\gamma}_g g_t, \tag{C8}$$

with

$$\tilde{\gamma}_y = \frac{(\gamma_\pi \psi_y \psi_\pi b_\pi^\pi (1+h) + \gamma_\pi h(\omega+\psi_y\beta) + (h+1)\gamma_y^2 \left(1-b_\pi^y\psi_y\right))}{((1-b_\pi^y\psi_y)(1+h) - \gamma_\pi(\omega+\beta\psi_y))}, \tag{C9}$$

$$\tilde{\gamma}_\pi = \frac{\gamma_\pi(1+h)(\gamma_y(1-b_\pi^y\psi_y) + b_\pi^\pi\psi_\pi^2)}{((1-b_\pi^y\psi_y)(1+h) - \gamma_\pi(\omega+\beta\psi_y))}, \tag{C10}$$

$$\tilde{\gamma}_u = \frac{(1+h)(1-b_\pi^y\psi_y)(\gamma_y\gamma_u+\gamma_\pi)}{((1-b_\pi^y\psi_y)(1+h) - \gamma_\pi(\omega+\beta\psi_y))}, \tag{C11}$$

$$\tilde{\gamma}_g = \frac{(\gamma_y\gamma_g(1-b_\pi^y\psi_y)(1+h) + \frac{\gamma_\pi(1-h)}{\sigma}(\beta\psi_y+\omega))}{((1-b_\pi^y\psi_y)(1+h) - \gamma_\pi(\omega+\beta\psi_y))}, \tag{C12}$$

which will be the expert advice.

To calculate the coefficients, we plug the (C9)-(C12) into (C3). The coefficients of the experts' rule will be the solution to the following equations and are functions of agents' PLM.

$$\gamma_\pi = \frac{b_\pi^y\psi_\pi^2}{(1-b_\pi^y\psi_y)} + \frac{\gamma_\pi(\gamma_y(1-b_\pi^y\psi_y) + b_\pi^\pi\psi_\pi^2)}{(1-b_\pi^y\psi_y)((1-b_\pi^y\psi_y)(1+h) - \gamma_\pi(\omega+\beta\psi_y))}, \tag{C13}$$

$$\begin{aligned}
\gamma_y &= \frac{b_\pi^y\psi_\pi\psi_y(1+h)+h}{(1-b_\pi^y\psi_y)(1+h)} \\
&+ \frac{1}{(1-b_\pi^y\psi_y)(1+h)} \frac{(\gamma_\pi \psi_y \psi_\pi b_\pi^\pi (1+h) + \gamma_\pi h(\omega+\psi_y\beta) + (h+1)\gamma_y^2 \left(1-b_\pi^y\psi_y\right))}{((1-b_\pi^y\psi_y)(1+h) - \gamma_\pi(\omega+\beta\psi_y))},
\end{aligned} \tag{C14}$$

$$\gamma_g = \frac{h-1}{\sigma\left((h+1)\left(b_\pi^y\psi_y-1\right) + \gamma_\pi\left(\beta\psi_y+\omega\right) + \gamma_y\right)}, \tag{C15}$$

$$\gamma_u = \frac{\gamma_\pi}{(h+1)\left(1-b_\pi^y\psi_y\right) - \gamma_\pi\left(\beta\psi_y+\omega\right) - \gamma_y}. \tag{C16}$$

[15] That is, taking into account agents' PLM for inflation.

Appendix D: Rational Expectations MSV

Under REE MSV, the perceived law of motion for the system in (B6) is:

$$\begin{aligned} \begin{bmatrix} \pi_t \\ y_t \end{bmatrix} &= \Omega + \bar{C} \begin{bmatrix} \pi_{t-1} \\ y_{t-1} \end{bmatrix} + \Gamma \begin{bmatrix} u_t \\ g_t \end{bmatrix}, && \text{(D1)} \\ E_t \begin{bmatrix} \pi_{t+1} \\ y_{t+1} \end{bmatrix} &= \Omega + \bar{C} \begin{bmatrix} \pi_t \\ y_t \end{bmatrix} = \Omega + \bar{C}\Omega + \bar{C}^2 \begin{bmatrix} \pi_{t-1} \\ y_{t-1} \end{bmatrix} + \bar{C}\Gamma \begin{bmatrix} u_t \\ g_t \end{bmatrix}. && \text{(D2)} \end{aligned}$$

Plugging the PLM into (B6):

$$\begin{bmatrix} \pi_t \\ y_t \end{bmatrix} = A(I+\bar{C})\Omega + \left[A\left(\bar{C}\right)^2 + C\right] \begin{bmatrix} \pi_{t-1} \\ y_{t-1} \end{bmatrix} + \left[A\bar{C}\Gamma + B\right] \begin{bmatrix} u_t \\ g_t \end{bmatrix}. \quad \text{(D3)}$$

Using the method of undetermined coefficients, we can solve for the PLM coefficients from:

$$\begin{aligned} \bar{C} &= A\bar{C}^2 + C = 0, && \text{(D4)} \\ \Gamma &= B + A\bar{C}\Gamma, && \text{(D5)} \\ \Omega &= A(I+\bar{C})\Omega. && \text{(D6)} \end{aligned}$$

Generically, the eigenvalues of $A(I+\bar{C})$ are not equal to unity, and therefore the solution for the constant vector Ω is a zero vector.

The MSV coefficients are the solution for the following system

$$\begin{aligned} c_\pi^y &= \frac{\omega h}{1+h} + (\beta - \frac{\omega(1-h)(\phi_\pi - 1)}{(1+h)\sigma}) c_\pi^y c_y^y + \frac{\omega}{1+h} c_y^{y2}, && \text{(D7)} \\ c_y^y &= \frac{h}{1+h} - \frac{(1-h)(\phi_\pi - 1)}{(1+h)\sigma} c_\pi^y c_y^y + \frac{1}{1+h} c_y^{y2}. && \text{(D8)} \end{aligned}$$

While the first equation is quadratic, the second is cubic, with the determinant changing signs depending on the central bank's reaction function. Thus we have 3 possible solutions. Following McCallum (1983), we choose the solution for c_π^y which goes to zero when $\omega = 0$ and we impose the condition $0 < c_y^y < 1$.

Multiplying the second equation by ω and subtracting from the first, we get an expression for c_π^y:

$$c_\pi^y = \frac{\omega c_y^y}{1 - \beta c_y^y}. \quad \text{(D9)}$$

For $0 < c_y^y < 1$ it follows form (D9), that $c_\pi^y > 0$.

It is instructive to consider c_y^y as a function of c_π^y solving the second equation. The two solutions will be:

$$\begin{aligned} c_y^y &= \tfrac{1+h}{2} \left(1 + \tfrac{(1-h)(\phi_\pi - 1)}{(1+h)\sigma} c_\pi^y + \sqrt{(1 + \tfrac{(1-h)(\phi_\pi - 1)}{(1+h)\sigma} c_\pi^y)^2 - \tfrac{4h}{(1+h)^2}} \right), \\ c_y^y &= \tfrac{1+h}{2} \left(1 + \tfrac{(1-h)(\phi_\pi - 1)}{(1+h)\sigma} c_\pi^y - \sqrt{(1 + \tfrac{(1-h)(\phi_\pi - 1)}{(1+h)\sigma} c_\pi^y)^2 - \tfrac{4h}{(1+h)^2}} \right). \end{aligned}$$

Clearly, for $\phi_\pi > 1$, the first solution is larger than unity. The second solution for $\phi_\pi > 1$ is larger than zero and smaller than $\frac{1+h}{2}$:

$$0 < \tfrac{1+h}{2}\left(1 + \tfrac{(1-h)(\phi_\pi - 1)}{(1+h)\sigma}c_\pi^y - \sqrt{(1 + \tfrac{(1-h)(\phi_\pi-1)}{(1+h)\sigma}c_\pi^y)^2 - \tfrac{4h}{(1+h)^2}}\right) < \tfrac{1+h}{2},$$
$$\Rightarrow \tfrac{(1-h)(\phi_\pi-1)}{(1+h)\sigma}c_\pi^y - \sqrt{(1 + \tfrac{(1-h)(\phi_\pi-1)}{(1+h)\sigma}c_\pi^y)^2 - \tfrac{4h}{(1+h)^2}} < 0. \tag{D10}$$

We define the solution as $\bar{C} = \begin{bmatrix} 0 & c_\pi^y \\ 0 & c_y^y \end{bmatrix}$, $\Gamma = \begin{bmatrix} 1 & \gamma_\pi^y \\ 0 & \gamma_y^y \end{bmatrix}$, $\Omega = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

D.1 E-Stability

To study E-Stability for our model, we write T-mapping as a system of equations:

$$c_\pi^y \quad -> \quad \frac{\omega h}{1+h} + (\beta - \frac{\omega(1-h)(\phi_\pi - 1)}{(1+h)\sigma})c_\pi^y c_y^y + \frac{\omega}{1+h}c_y^{y2}, \tag{D11}$$
$$c_y^y \quad -> \quad \frac{h}{1+h} - \frac{(1-h)(\phi_\pi - 1)}{(1+h)\sigma}c_\pi^y c_y^y + \frac{1}{1+h}c_y^{y2}, \tag{D12}$$

with the Jacobian

$$\begin{bmatrix} (\beta - \frac{\omega(1-h)(\phi_\pi-1)}{(1+h)\sigma})c_y^y - 1 & (\beta - \frac{\omega(1-h)(\phi_\pi-1)}{(1+h)\sigma})c_\pi^y + \frac{2\omega}{1+h}c_y^y \\ -\frac{(1-h)(\phi_\pi-1)}{(1+h)\sigma}c_y^y & -\frac{(1-h)(\phi_\pi-1)}{(1+h)\sigma}c_\pi^y - 1 + \frac{2}{1+h}c_y^y \end{bmatrix} = \begin{bmatrix} J11 & J12 \\ J21 & J22 \end{bmatrix}. \tag{D13}$$

As long as the Taylor principle $\phi_\pi > 1$ is satisfied, $J11$ and $J21$ are negative. For reasonable ϕ_π, $J12 > 0$.[16]

For $J22 < 0$, the following conditions must hold: $c_y^y \leq (1+h)/2$ and $\phi_\pi > 1$. As shown in D10 for $\phi_\pi > 1$, $c_y^y < (1+h)/2$.

The discriminant (product of eigenvalues) of (D13) is then $-J12J21 + J11J22$ and the trace (sum of eigenvalues) is $J11 + J22$. The discriminant is positive and the trace is negative.

Thus, the sufficient condition for both eigenvalues to be negative is the Taylor principle $\phi_\pi > 1$.

To study **strong E-Stability**, we allow for arbitrary matrix $\bar{C}$ with the following coefficients:

$$\bar{C} = \begin{bmatrix} v & c_\pi^y \\ w & c_y^y \end{bmatrix}. \tag{D14}$$

The part of T-mapping responsible for $\bar{C}$ is modified:

$$\begin{bmatrix} b_\pi^\pi & \frac{\omega}{1+h} \\ b_\pi^y & \frac{1}{1+h} \end{bmatrix} \begin{bmatrix} v & c_\pi^y \\ w & c_y^y \end{bmatrix} \begin{bmatrix} v & c_\pi^y \\ w & c_\pi^y \end{bmatrix} + \begin{bmatrix} 0 & \frac{\omega h}{1+h} \\ 0 & \frac{h}{1+h} \end{bmatrix} =$$
$$\begin{bmatrix} b_\pi^\pi(v^2 + wc_\pi^y) + \frac{\omega w}{1+h}(v + c_y^y) & b_\pi^\pi(v + c_y^y)c_\pi^y + \frac{\omega}{1+h}(wc_\pi^y + (c_y^y)^2) + \frac{\omega h}{1+h} \\ b_\pi^y(v^2 + wc_\pi^y) + \frac{w}{1+h}(v + c_y^y) & b_\pi^y(v + c_y^y)c_\pi^y + \frac{h}{1+h}(wc_\pi^y + (c_y^y)^2) + \frac{h}{1+h} \end{bmatrix}, \tag{D15}$$

[16] To make $J12$ negative, the reaction of monetary policy to inflation should be stronger than empirically plausible.

where the elements of matrices A and C are given in (B6).

The part of the Jacobian responsible for these coefficients becomes:

$$J=\begin{bmatrix} 2b^{\pi}_{\pi}\bar{v}+\frac{\omega}{1+h}\bar{w}-1 & b^{\pi}_{\pi}\bar{c}^{y}_{\pi}+\frac{\omega}{1+h}(\bar{v}+\bar{c}^{y}_{y}) & b^{\pi}_{\pi}\bar{w} & \frac{\omega}{1+h}\bar{w} \\ 2b^{y}_{\pi}\bar{v}+\frac{1}{1+h}\bar{w} & b^{y}_{\pi}\bar{c}^{y}_{\pi}+\frac{1}{1+h}(\bar{v}+\bar{c}^{y}_{y})-1 & b^{y}_{\pi}\bar{w} & \frac{1}{1+h}\bar{w} \\ b^{\pi}_{\pi}\bar{c}^{y}_{\pi} & \frac{\omega}{1+h}\bar{c}^{y}_{\pi} & b^{\pi}_{\pi}(\bar{v}+\bar{c}^{y}_{y})+\frac{\omega}{1+h}\bar{w}-1 & b^{\pi}_{\pi}\bar{c}^{y}_{\pi}+2\frac{\omega}{1+h}\bar{c}^{y}_{y} \\ b^{y}_{\pi}\bar{c}^{y}_{\pi} & \frac{1}{1+h}\bar{c}^{y}_{\pi} & b^{y}_{\pi}\bar{c}^{y}_{y} & b^{y}_{\pi}\bar{c}^{y}_{\pi}+2\frac{1}{1+h}\bar{c}^{y}_{y}-1 \end{bmatrix}.$$

The condition for strong E-Stability is

$$Eig(J)<0. \tag{D16}$$

With $\bar{v}=\bar{w}=0$, the Jacobian becomes:

$$J=\begin{bmatrix} -1 & b^{\pi}_{\pi}\bar{c}^{y}_{\pi}+\frac{\omega}{1+h}\bar{c}^{y}_{y} & 0 & 0 \\ 0 & b^{y}_{\pi}\bar{c}^{y}_{\pi}+\frac{1}{1+h}\bar{c}^{y}_{y}-1 & 0 & 0 \\ b^{\pi}_{\pi}\bar{c}^{y}_{\pi} & \frac{\omega}{1+h}\bar{c}^{y}_{\pi} & b^{\pi}_{\pi}\bar{c}^{y}_{y}-1 & b^{\pi}_{\pi}\bar{c}^{y}_{\pi}+2\frac{\omega}{1+h}\bar{c}^{y}_{y} \\ b^{y}_{\pi}\bar{c}^{y}_{\pi} & \frac{1}{1+h}\bar{c}^{y}_{\pi} & b^{y}_{\pi}\bar{c}^{y}_{y} & b^{y}_{\pi}\bar{c}^{y}_{\pi}+2\frac{1}{1+h}\bar{c}^{y}_{y}-1 \end{bmatrix}, \tag{D17}$$

where the lower 2×2 block is the same as (D13) for weak E-stability. In addition to two eigenvalues identical to those of (D13), this matrix has two more eigenvalues: -1 and $b^{y}_{\pi}\bar{c}^{y}_{\pi}+\frac{1}{1+h}\bar{c}^{y}_{y}-1$. For the second extra eigenvalue to be negative, $b^{y}_{\pi}\bar{c}^{y}_{\pi}+\frac{1}{1+h}\bar{c}^{y}_{y}-1=J22-1<J22$ must hold.

Thus, the sufficient condition for strong E-stability is satisfied as long as the sufficient condition for weak E-stability is satisfied.

Appendix E: Restricted Perception Equilibrium

E.1 Definition of RPE and ALM Coefficients

We focus on the RPE M_{π} and derive the conditions for its existence below.

The agents' inflation expectations with the M_{π} forecasting rule (33) are formulated as follows:

$$\pi_{t+1} \quad = \quad \alpha^{\pi}_{\pi}+\beta^{\pi}_{\pi}\pi_t=\alpha^{\pi}_{\pi}(1+\beta^{\pi}_{\pi})+(\beta^{\pi}_{\pi})^2\pi_{t-1}. \tag{E1}$$

When we plug the above M_{π} into the model in (27) using an expert forecast for output (C8), we get the inflation ALM:

$$\begin{aligned}\pi_t &= b^{\pi}_{\pi}\alpha^{\pi}_{\pi}(1+\beta^{\pi}_{\pi})+\left(b^{\pi}_{\pi}(\beta^{\pi}_{\pi})^2+\frac{\tilde{\gamma}_{\pi}\omega}{1+h}\right)\pi_{t-1}+\frac{\omega(h+\tilde{\gamma}_y)}{1+h}y_{t-1} \\ &+\frac{\omega((1-h)+\sigma\tilde{\gamma}_g)}{(1+h)\sigma}g_t+(\frac{\omega\tilde{\gamma}_u}{1+h}+1)u_t= \\ &=\bar{a}_{\pi}+\bar{b}_{\pi}\pi_{t-1}+\bar{c}_{\pi}y_{t-1}+\bar{\eta}^{g}_{\pi}g_t+\bar{\eta}^{u}_{\pi}u_t,\end{aligned} \tag{E2}$$

with $b_{\pi}=\beta-\frac{\omega(\phi_{\pi}-1)(1-h)}{\sigma(1+h)}$, and $\bar{x}_{\pi}$ being the coefficients in inflation ALM.

Similarly, the ALM for the output gap is:

$$y_t = b^y_\pi(\alpha^\pi_\pi(1+\beta^\pi_\pi)) + \left(b^y_\pi(\beta^\pi_\pi)^2 + \frac{\tilde{\gamma}_\pi}{1+h}\right)\pi_{t-1} + \frac{h+\tilde{\gamma}_y}{1+h}y_{t-1} + \frac{1-h+\sigma\tilde{\gamma}_g}{(1+h)\sigma}g_t +$$
$$\frac{\tilde{\gamma}_u}{1+h}u_t = \bar{a}_y + \bar{b}_y\pi_{t-1} + \bar{c}_y y_{t-1} + \bar{\eta}^g_y g_t + \bar{\eta}^u_y u_t, \quad \text{(E3)}$$

with $b^y_\pi = -\frac{1-h}{(1+h)\sigma}(\phi_\pi - 1)$, and $\bar{x}_y$ are output gap ALM coefficients.

When we focus on a converged RPE M_π, we can plug $\psi_y = 0$ and $\psi_\pi = \beta^\pi_\pi$ into expert forecast coefficients. The solution for γ and $\tilde{\gamma}$ is the solution to the following equations:

$$\gamma_\pi = b^y_\pi(\beta^\pi_\pi)^2 + \frac{\gamma_\pi(\gamma_y + b^\pi_\pi(\beta^\pi_\pi)^2)}{1+h-\gamma_\pi\omega}, \quad \text{(E4)}$$

$$\gamma_y = \frac{h+\gamma_y^2}{1+h-\gamma_\pi\omega}, \quad \text{(E5)}$$

$$\gamma_g = \frac{1-h}{\sigma\left(1+h-\gamma_\pi\omega-\gamma_y\right)}, \quad \text{(E6)}$$

$$\gamma_u = \frac{\gamma_\pi}{((1+h)-\gamma_\pi\omega)-\gamma_y}; \quad \text{(E7)}$$

and

$$\tilde{\gamma}_y = \frac{\gamma_\pi h\omega + (1+h)\gamma_y^2}{1+h-\gamma_\pi\omega}, \quad \text{(E8)}$$

$$\tilde{\gamma}_\pi = \frac{\gamma_\pi(1+h)(\gamma_y + b^\pi_\pi(\beta^\pi_\pi)^2)}{1+h-\gamma_\pi\omega}, \quad \text{(E9)}$$

$$\tilde{\gamma}_u = \frac{(1+h)\gamma_\pi}{((1+h)-\gamma_\pi\omega-\gamma_y)}, \quad \text{(E10)}$$

$$\tilde{\gamma}_g = (1-h)\frac{\gamma_y + \omega\gamma_\pi}{(\sigma\left(1+h-\gamma_\pi\omega-\gamma_y\right))}. \quad \text{(E11)}$$

It is instructive to examine the coefficients. An economically meaningful coefficient on lagged output is $0 < \gamma_y < 1$. It follows from $(E5)$ that $\omega\gamma_\pi \leq 1+h-2\sqrt{h}$ and $\gamma_\pi < (1-\gamma_y^2)/\omega$; and $0 < \tilde{\gamma}_y < 1$.

Plugging (E4):(E11) into (E2):(E3), we obtain ALM coefficients for output and inflation.

$$\bar{c}_\pi = \omega\frac{(h+\tilde{\gamma}_y)}{1+h} = \omega\gamma_y, \quad \text{(E12)}$$

$$\bar{b}_\pi = b^\pi_\pi(\beta^\pi_\pi)^2 + \frac{\tilde{\gamma}_\pi\omega}{1+h} = \omega\gamma_\pi + \beta(\beta^\pi_\pi)^2, \quad \text{(E13)}$$

$$\bar{c}_y = \frac{h+\tilde{\gamma}_y}{1+h} = \gamma_y, \quad \text{(E14)}$$

$$\bar{b}_y = b^y_\pi(\beta^\pi_\pi)^2 + \frac{\tilde{\gamma}_\pi}{1+h} = \gamma_\pi. \quad \text{(E15)}$$

Now, the ALM coefficients for shock processes:

$$\bar{\eta}^g_\pi = \frac{(1-h)\omega}{(1+h-\gamma_\pi\omega-\gamma_y)\sigma}, \quad \text{(E16)}$$

$$\bar{\eta}_{\pi}^{u} = \frac{(1+h)-\gamma_y}{((1+h)-\gamma_\pi\omega-\gamma_y)}, \tag{E17}$$

$$\bar{\eta}_{y}^{g} = \frac{1-h}{(1+h-\gamma_\pi\omega-\gamma_y)\sigma}, \tag{E18}$$

$$\bar{\eta}_{y}^{u} = \frac{\gamma_\pi}{1+h-\gamma_\pi\omega-\gamma_y}. \tag{E19}$$

E.2 RPE Beliefs

We treat the agents as econometricians, who learn the coefficients from running regressions of the corresponding PLMs. Denoting the covariance between inflation and output $Cov(\pi, y) \equiv \sigma_{\pi y}$, and the variances of output and inflation as σ_y^2 and σ_π^2 respectively, we can derive the coefficients for the M_π forecasting rule:

$$\begin{aligned}
\beta_\pi^\pi &= \frac{Cov(\pi_t, \pi_{t-1})}{Var(\pi_{t-1})} = \frac{Cov(\bar{b}_\pi \pi_{t-1} + \bar{c}_\pi y_{t-1}, \pi_{t-1})}{Var(\pi_{t-1})} = \\
&= \bar{b}_\pi + \bar{c}_\pi \frac{\sigma_{\pi y}}{\sigma_\pi^2}, && \text{(E20)}\\
\alpha_\pi^\pi &= (1-\beta_\pi^\pi)\bar{\pi}. && \text{(E21)}
\end{aligned}$$

For the M_y rule, the regression coefficients are computed with the inflation ALM given by M_π in (E2):

$$\begin{aligned}
\beta_y^y &= \frac{Cov(\pi_t, y_{t-1})}{Var(y_{t-1})} = \frac{Cov\left(\bar{b}_\pi \pi_{t-1} + \bar{c}_\pi y_{t-1}, y_{t-1}\right)}{Var(y_{t-1})} = \\
&= \bar{b}_\pi \frac{\sigma_{\pi y}}{\sigma_y^2} + \bar{c}_\pi, && \text{(E22)}\\
\alpha_y^y &= (1-\beta_y^y)\bar{\pi}. && \text{(E23)}
\end{aligned}$$

E.2.1 Proof of Proposition 5

Existence of RPE.

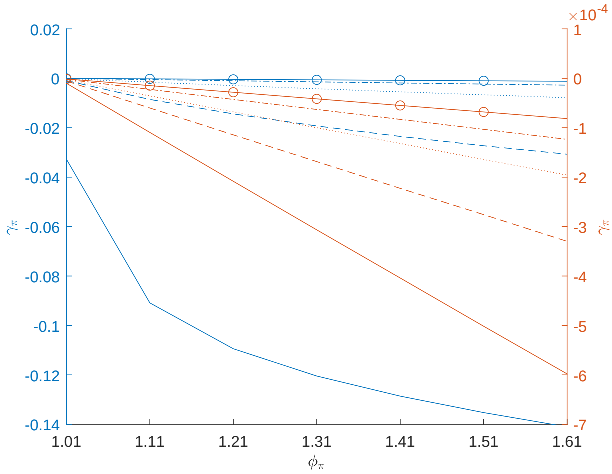
For the M_π to exist, there must exist a β_π^π, which is a solution for (E20). Audzei and Slobodyan (2022) has shown that there exists a unique solution for (E20), as long as the following matrix D is stable:

$$D = \begin{bmatrix} \bar{b}_\pi & \bar{c}_\pi \\ \bar{b}_y & \bar{c}_y \end{bmatrix} = \begin{bmatrix} \omega\gamma_\pi + \beta(\beta_\pi^\pi)^2 & \omega\gamma_y \\ \gamma_\pi & \gamma_y \end{bmatrix}. \tag{E24}$$

Determinant and trace are given by the following expressions:

$$\begin{aligned}
det(D) &= -\gamma_\pi\omega\gamma_y + (\omega\gamma_\pi + \beta(\beta_\pi^\pi)^2)\gamma_y = (\beta_\pi^\pi)^2\gamma_y\beta && \text{(E25)}\\
tr(D) &= \omega\gamma_\pi + \beta(\beta_\pi^\pi)^2 + \gamma_y. && \text{(E26)}
\end{aligned}$$

Figure E1: γ_π ***for Different Values of Shocks' Relative Variance***



Note: The figure is drawn for r – relative standard deviations of inflationary shocks, in the range: [0.1:0.1:0.5] left axis and [0.6:0.1:1] right axis.

For the matrix to be stable, we use the following conditions (see Audzei and Slobodyan 2022, Appendix B, for details):

$$det(D) < 1, \tag{E27}$$

$$det(D) > tr(D) - 1, \tag{E28}$$

$$det(D) > -tr(D) - 1. \tag{E29}$$

The condition in (E27) is satisfied as $(\beta_\pi^\pi)^2\gamma_y\beta < 1$.

To prove that the (E28) is satisfied, we combine (E25) and (E26) and rewrite them as:

$$\omega\gamma_\pi + \beta(\beta_\pi^\pi)^2 + \gamma_y < (\beta_\pi^\pi)^2\gamma_y\beta + 1. \tag{E30}$$

For $\gamma_\pi \leq 0$, given that $\beta_\pi^\pi < 1$, $\beta < 1$, and $\gamma_y < 1$, it is straightforward to show that $\beta(\beta_\pi^\pi)^2 + \gamma_y < 1 + \beta(\beta_\pi^\pi)^2\gamma_y$.

Values of $\gamma_\pi > 0$ are not economically meaningful: further, during our simulations there was no stable solution with $\gamma_\pi > 0$ for our parametrization. In Figure E1, we plot the solutions for γ_π as a function of monetary policy response to inflation and relative volatility of mark-up shocks to show that the solution for γ_π is always below zero.

The condition (E29) is satisfied as long as $(\beta_\pi^\pi)^2\gamma_y\beta > 0$ and $\gamma_\pi < 0$.

Thus, the matrix D is stable and the unique RPE solution exists.
Mapping and Variance-Covariance Matrix.

To calculate observed average inflation, rewrite M_π ALM as

$$\left[I - \begin{pmatrix} \bar{b}_\pi & \bar{c}_\pi \\ \bar{b}_y & \bar{c}_y \end{pmatrix}\right] \begin{bmatrix} \bar{\pi} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} \bar{a}_\pi \\ \bar{a}_y \end{bmatrix}, \tag{E31}$$

$$\Rightarrow \begin{bmatrix} \bar{\pi} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} \frac{\bar{a}_y\bar{c}_\pi + \bar{a}_\pi(1-\bar{c}_y)}{1-(\bar{b}_\pi+\bar{c}_y)+(\bar{b}_\pi\bar{c}_y-\bar{b}_y\bar{c}_\pi)} \\ \frac{\bar{a}_y-\bar{a}_y\bar{b}_\pi+\bar{a}_\pi\bar{b}_y}{1-(\bar{b}_\pi+\bar{c}_y)+(\bar{b}_\pi\bar{c}_y-\bar{b}_y\bar{c}_\pi)} \end{bmatrix}. \tag{E32}$$

To calculate the variances-covariance matrix of the ALM, re-write:

$$\begin{aligned} \Sigma = \begin{bmatrix} \sigma_\pi^2 & \sigma_{\pi y} \\ \sigma_{\pi y} & \sigma_y^2 \end{bmatrix} &= \begin{bmatrix} \bar{b}_\pi & \bar{c}_\pi \\ \bar{b}_y & \bar{c}_y \end{bmatrix} \begin{bmatrix} \sigma_\pi^2 & \sigma_{\pi y} \\ \sigma_{\pi y} & \sigma_y^2 \end{bmatrix} \begin{bmatrix} \bar{b}_\pi & \bar{b}_y \\ \bar{c}_\pi & \bar{c}_y \end{bmatrix} + \\ &+ \begin{bmatrix} \bar{\eta}_\pi^u & \bar{\eta}_\pi^g \\ \bar{\eta}_y^u & \bar{\eta}_y^g \end{bmatrix} \begin{bmatrix} \sigma_u^2 & 0 \\ 0 & \sigma_g^2 \end{bmatrix} \begin{bmatrix} \bar{\eta}_\pi^u & \bar{\eta}_y^u \\ \bar{\eta}_\pi^g & \bar{\eta}_y^g \end{bmatrix} =, \\ = \begin{bmatrix} (\bar{b}_\pi)^2\sigma_\pi^2 + 2\bar{b}_\pi\bar{c}_\pi\sigma_{\pi y} + (\bar{c}_\pi)^2\sigma_y^2 & \bar{b}_\pi\bar{b}_y\sigma_\pi^2 + (\bar{c}_\pi\bar{b}_y + \bar{b}_\pi\bar{c}_y)\sigma_{\pi y} + \bar{c}_\pi\bar{c}_y\sigma_y^2 \\ \bar{b}_\pi\bar{b}_y\sigma_\pi^2 + (\bar{c}_\pi\bar{b}_y + \bar{b}_\pi\bar{c}_y)\sigma_{\pi y} + \bar{c}_\pi\bar{c}_y\sigma_y^2 & (\bar{c}_y)^2\sigma_y^2 + (\bar{b}_y)^2\sigma_\pi^2 + 2\bar{b}_y\bar{c}_y\sigma_{\pi y} \end{bmatrix} &+ \\ + \begin{bmatrix} (\bar{\eta}_\pi^u)^2\sigma_u^2 + (\bar{\eta}_\pi^g)^2\sigma_g^2 & \bar{\eta}_\pi^g\bar{\eta}_y^g\sigma_g^2 + \bar{\eta}_\pi^u\bar{\eta}_y^u\sigma_u^2 \\ \bar{\eta}_\pi^g\bar{\eta}_y^g\sigma_g^2 + \bar{\eta}_\pi^u\bar{\eta}_y^u\sigma_u^2 & (\bar{\eta}_y^g)^2\sigma_g^2 + (\bar{\eta}_y^u)^2\sigma_u^2 \end{bmatrix}&. \end{aligned} \tag{E33}$$

The elements of the variance-covariance matrix are the solution for the following equations:

$$\sigma_\pi^2 = (\bar{b}_\pi)^2\sigma_\pi^2 + 2\bar{b}_\pi\bar{c}_\pi\sigma_{\pi y} + (\bar{c}_\pi)^2\sigma_y^2 + (\bar{\eta}_\pi^u)^2\sigma_u^2 + (\bar{\eta}_\pi^g)^2\sigma_g^2, \tag{E34}$$

$$\sigma_{\pi y} = \bar{b}_\pi\bar{b}_y\sigma_\pi^2 + (\bar{c}_\pi\bar{b}_y + \bar{b}_\pi\bar{c}_y)\sigma_{\pi y} + \bar{c}_\pi\bar{c}_y\sigma_y^2 + \bar{\eta}_\pi^g\bar{\eta}_y^g\sigma_g^2 + \bar{\eta}_\pi^u\bar{\eta}_y^u\sigma_u^2, \tag{E35}$$

$$\sigma_y^2 = (\bar{c}_y)^2\sigma_y^2 + (\bar{b}_y)^2\sigma_\pi^2 + 2\bar{b}_y\bar{c}_y\sigma_{\pi y} + (\bar{\eta}_y^g)^2\sigma_g^2 + (\bar{\eta}_y^u)^2\sigma_u^2. \tag{E36}$$

E-Stability From (E20) and (E21), the T-map for the RPE M_π is:

$$\beta_\pi^\pi \to \bar{b}_\pi + \bar{c}_\pi \frac{\sigma_{\pi y}}{\sigma_\pi^2}, \tag{E37}$$

$$\alpha_\pi^\pi \to (1-\beta_\pi^\pi)\bar{\pi}. \tag{E38}$$

For M_π to be E-stable, eigenvalues of the following matrix should be negative:

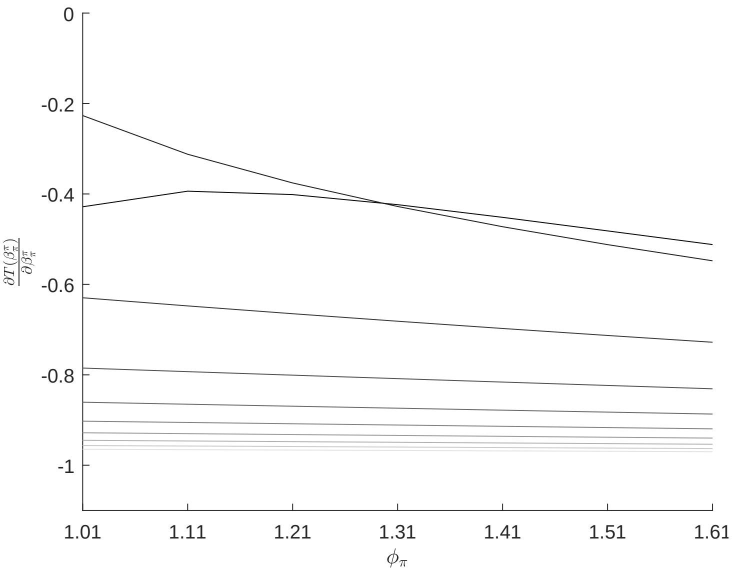
$$\begin{aligned} Eig \begin{bmatrix} (1-\beta_\pi^\pi)\frac{\partial(\bar{\pi})}{\partial\alpha_\pi^\pi} - 1 & \frac{\partial((1-\beta_\pi^\pi)\bar{\pi})}{\partial\beta_\pi^\pi} \\ 0 & \frac{\partial\left[\bar{b}_\pi + \bar{c}_\pi\frac{\sigma_{\pi y}}{\sigma_\pi^2}\right]}{\partial\beta_\pi^\pi} - 1 \end{bmatrix} &= \\ \begin{bmatrix} (1-\beta_\pi^\pi)\frac{\partial(\bar{\pi})}{\partial\alpha_\pi^\pi} - 1 & \frac{\partial\left[\bar{b}_\pi + \bar{c}_\pi\frac{\sigma_{\pi y}}{\sigma_\pi^2}\right]}{\partial\beta_\pi^\pi} - 1 \end{bmatrix} &< 0. \end{aligned} \tag{E39}$$

In the text we have assumed that $\bar{\pi} = 0$; in this case the first eigenvalue is negative. We plot the second eigenvalue for the parameter range $r = \sigma_u/\sigma_g \in (0:1]$ for $\phi_\pi > 1$. As both eigenvalues are negative for the considered parameter range, we conclude that M_π is E-Stable.

Better forecasting performance.

For better forecasting performance of M_π relative to M_y we consider the mean squared forecast errors criterion, given that agents have previously selected M_π and test alternative models given M_π

Figure E2: E-Stability



Note: The figure is drawn for r – relative standard deviations of inflationary shocks in the range $[0.1 : 0.1 : 1]$. The darker colours correspond to smaller r

ALM in (E2). It is convenient to denote the composite of shocks as $\mu_t \equiv \bar{\eta}_\pi^g g_t + \bar{\eta}_\pi^u u_t$. We start with the mean forecast error of M_π. The forecast error of M_π is the difference between the forecast and actual inflation:

$$\begin{aligned} : \quad & e_t^\pi = (\beta_\pi^\pi - \bar{b}_\pi)\pi_{t-1} - \bar{c}_\pi y_{t-1} + \mu_t \\ = \quad & (\bar{b}_\pi + \bar{c}_\pi \frac{\sigma_{\pi y}}{\sigma_\pi^2} - \bar{b}_\pi)\pi_{t-1} - \bar{c}_\pi y_{t-1} - \mu_t \\ = \quad & \bar{c}_\pi \frac{\sigma_{\pi y}}{\sigma_\pi^2} \pi_{t-1} - \bar{c}_\pi y_{t-1} + \mu_t, \end{aligned} \tag{E40}$$

$$\begin{aligned} : \quad & MSFE_\pi = E_t(e_t^\pi)^2 = E_t[\bar{c}_\pi \frac{\sigma_{\pi y}}{\sigma_\pi^2}(\pi_{t-1}) - \bar{c}_\pi(y_{t-1}) + \mu_t]^2 \\ : \quad & = E_t[\bar{c}_\pi^2 (\frac{\sigma_{\pi y}}{\sigma_\pi^2})^2 (\pi_{t-1})^2 - 2\bar{c}_\pi R \bar{c}_\pi (\pi_{t-1})(y_{t-1}) + \bar{c}_\pi^2 (y_{t-1})^2 + \mu^2] \\ : \quad & = \bar{c}_\pi^2 \sigma_y^2 (1 - \frac{\sigma_{\pi y}^2}{\sigma_\pi^2 \sigma_y^2}) + \sigma_\mu^2. \end{aligned} \tag{E41}$$

Similarly, the forecast error of M_y is:

$$\begin{aligned} : \quad & e_t^y = \left(c_y^y - \bar{c}_\pi\right) y_{t-1} - \bar{b}_\pi \pi_{t-1} + \mu_t = \\ : \quad & = \bar{b}_\pi \sigma_{\pi y} y_{t-1} - \bar{b}_\pi \pi_{t-1} + \mu_t = \\ : \quad & MSFE_y = E[\bar{b}_\pi [\frac{\sigma_{\pi y}}{\sigma_y^2}(y_{t-1}) - (\pi_{t-1})]^2 + \mu^2] \end{aligned} \tag{E42}$$

$$: \quad = \bar{b}_\pi^2 \sigma_\pi^2 [1 - \frac{\sigma_{\pi y}^2}{\sigma_y^2 \sigma_\pi^2}] + \sigma_\mu^2. \tag{E43}$$

We are looking for the conditions under which $MSFE_\pi < MSFE_y$. Then, the criterion is simply:

$$\bar{c}_\pi^2 \sigma_y^2 < \bar{b}_\pi^2 \sigma_\pi^2. \tag{E44}$$

Appendix F: Attention Weights Derivation for a New Keynesian Model

The selection of attention weights mimics the selection from the Fisher equation, except for the set of variables. Thus, the minimization problem is modified:

$$V \quad = E\left[\left(\bar{c}_\pi y_{t-1} + \bar{b}_\pi \pi_{t-1} - \gamma m_y y_{t-1} - \beta m_\pi \pi_{t-1}\right)^2\right] + \kappa\left(|m_y| + |m_\pi|\right), \tag{F1}$$

where the optimal attention vector m_π, m_y results in nine possible models with a functional form identical to the Fisher equation model. However, because both variables π and y are now endogenous, it will be impossible to analytically evaluate the boundaries between different forecasting rules being optimal.

Appendix G: Theoretical Foundations of Sliding Dynamics

The discussion in this section follows Jeffrey (2019), Ch. 2, and the concepts from Filippov (1988).

Suppose there is a vector ODE with a discontinuous flow,

$$\dot{\underline{\mathbf{x}}} = f(\underline{\mathbf{x}}, \lambda), \tag{G1}$$

so that at the boundary defined by $\mathscr{D} = \{\underline{\mathbf{x}} : \sigma(\underline{\mathbf{x}}) = 0\}$ there is a discontinuity of the function f. The surface $\mathscr{D}$ is called *discontinuity surface*. The *switching multiplier* λ could be selected so that $\lambda = sign(\sigma)$. Denote

$$\begin{aligned} f^+(\underline{\mathbf{x}}) \quad &: \quad = f(\underline{\mathbf{x}}; +1),\ \sigma(\underline{x}) > 0, \\ f^-(\underline{\mathbf{x}}) \quad &: \quad = f(\underline{\mathbf{x}}; -1),\ \sigma(\underline{x}) < 0. \end{aligned}$$

Then the time derivative of the flow above (below) the surface can be written as

$$\frac{d}{dt} = \frac{d\underline{\mathbf{x}}}{dt}\frac{d}{d\underline{\mathbf{x}}} = f^{\pm}\frac{d}{d\underline{\mathbf{x}}}.$$

The normal vector to $\mathscr{D}$ is defined as $\frac{d\sigma}{d\underline{\mathbf{x}}}$. Then, $f \cdot \frac{d\sigma}{d\underline{\mathbf{x}}} = \frac{d\underline{\mathbf{x}}}{dt}\frac{d\sigma}{d\underline{\mathbf{x}}} = \frac{d\sigma}{dt} = \dot{\sigma}$, so the projection of the vector f onto the normal vector to $\mathscr{D}$ gives the time derivative of σ.

The Lemma 2.1 of Jeffrey (2019) then states that if $f(\underline{\mathbf{x}}, \lambda)$ is continuous in λ and the components of $f^{\pm}(\underline{\mathbf{x}})$ normal to the boundary are in opposition to each other, there exists an intermediate value of λ, denoted $\lambda^{\$}$, $-1 \leq \lambda^{\$} \leq 1$, such that $f\left(\underline{\mathbf{x}}; \lambda^{\$}\right) \cdot \frac{d\sigma}{d\underline{\mathbf{x}}} = 0$. One can then further define solutions of the ODE (G1) that exist on the discontinuity surface, the *sliding* flow, so that

$$\begin{aligned} \dot{\underline{\mathbf{x}}} = f^{\$}(\underline{\mathbf{x}}) = f\left(\underline{\mathbf{x}}; \lambda^{\$}\right) \ \text{for}\ \sigma(\underline{x}) = 0, \\ f\left(\underline{\mathbf{x}}; \lambda^{\$}\right) \cdot \frac{d\sigma}{d\underline{\mathbf{x}}} = 0. \end{aligned} \tag{G2}$$

This flow's projection onto the normal to the boundary equals zero; thus $\sigma(\underline{\mathbf{x}}) = 0$ is preserved over time. However, there can be a non-zero projection to the subspace that is tangent to the boundary $\mathscr{D}$

at the point where it is reached by the original flow. This projection tangential to $\mathscr{D}$ gives rise to the sliding dynamics along the boundary.

If the components of $f^{\pm}(\underline{\mathbf{x}})$ normal to the boundary are pointing in the same direction, then a simple *crossing* of the boundary will happen, and no sliding along the boundary $\mathscr{D}$ will be observed. We check the opposing condition by computing the scalar product of the flows f^{+} and f^{-}, with a negative value signifying oppositely directed projections and thus the presence of sliding.

The easiest way to generate a function that is smooth in λ is to postulate that

$$\begin{aligned} &\dot{\underline{\mathbf{x}}} = f(\underline{\mathbf{x}},\lambda) = \frac{1}{2}(1+\lambda)\, f^{+}(\underline{\mathbf{x}}) + \frac{1}{2}(1-\lambda)\, f^{-}(\underline{\mathbf{x}}), \\ &\lambda = +1,\ \sigma(\underline{x}) > 0, \\ &\lambda = -1,\ \sigma(\underline{x}) < 0. \end{aligned} \tag{G3}$$

Then, one can define λ^{s} so that the projection of $\frac{1}{2}(1+\lambda^{s})\, f^{+}(\underline{\mathbf{x}}) + \frac{1}{2}(1-\lambda^{s})\, f^{-}(\underline{\mathbf{x}})$ on the normal to the boundary $\mathscr{D}$ is zero. The resulting flow then produces trajectories that slide along the boundary.

The construction above suggests the following simple algorithm for evaluating the trajectories of the approximating ODE that could involve sliding dynamics.

1. Trace the trajectory of the ODE solution until time T, stopping at $min(\tau, T)$, where τ is the first time the boundary V(0,0)=V(0,1) is reached.
2. If $\tau < T$, numerically compute the normal to the boundary V(0,0)=V(0,1) at the point at which it is achieved.
3. Check whether the scalar product of the projections of the flow $f^{\pm}(\underline{\mathbf{x}})$ onto the normal to the boundary is positive or negative.
4. If the product is positive, this is a simple crossing. Continue with Step 1, stopping at $min(\tau^{*}, T)$, where τ^{*} is the next time the boundary V(0,0)=V(0,1) is reached. Otherwise, switch to simulating the sliding ODE constructed as in (G2-G3) above, also until $min(\tau^{*}, T)$.
5. If $\tau^{*} < T$, repeat Step 3, otherwise end.

The algorithm described above could be thought of as a simplified version of Piiroinen and Kuznetsov (2008).

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