Working Paper Series ——— 3/2020

Growth-at-Risk: Bayesian Approach

Milan Szabo

Czech National Bank ——— Working Paper Series ——— 3/2020

CZECH NATIONAL BANK

www.cnb.cz

Reviewed by: Katarzyna Budnik (European Central Bank)
Tomáš Adam (Czech National Bank)

Project Coordinator: Martin Hodula

Growth-at-Risk: Bayesian Approach

Milan Szabo *

Abstract

The paper proposes a novel application of Bayesian quantile regression to forecast a full distribution of macroeconomic variables that can be linked to, for example, an official projection of the variable published by a central bank, or a forecast from a survey of professional forecasters. The approach is employed to estimate the popular *Growth-at-Risk*, which maps current financial and economic conditions to the distribution of future GDP growth, focusing mainly on downside risks. The results show that the linkage improves distribution forecasting and, thanks to the additional information obtained from the linkage, reduces overfitting and makes Growth-at-Risk models more operational for countries with short time series. Additional improvements in consistency around the official projection enhance the credibility of the results when communicated by the central bank. The method can also be used to derive asymmetric fan charts around the official projection not only for real GDP growth as examined in the paper, but also for unemployment or inflation.

Abstrakt

Článek předkládá originální aplikaci bayesovské kvantilové regrese pro predikci distribuce makroekonomických proměnných. Představená metoda umožňuje provázat odhad distribuce například s oficiální predikcí centrální banky nebo predikcemi získanými z dotazování analytiků. Přístup je představen skrze aplikaci na populární *Growth-at-Risk*, který promítá aktuální finanční a ekonomické podmínky do rozdělení budoucího růstu reálného HDP s důrazem na protirůstová rizika. Výsledky potvrzují, že provázání modelu zlepšuje jeho predikční schopnost. Zároveň je díky provázání rozšířena sada dostupných informací pro odhad modelu. Ta omezuje „přeučení" modelu a představuje velký benefit pro odhad s krátkými časovými řadami. Provázání dále zvyšuje konzistenci s oficiální predikcí a zvyšuje tak kredibilitu výsledků a jejich následné komunikace centrální bankou. Představená metoda pak může také sloužit k odhadu asymetrických vějířových grafů okolo oficiální predikce, a to nejen pro růst reálného HDP, který je modelovaný v článku, ale také nezaměstnanost nebo inflaci.

JEL Codes: C53, E27, E32, E44.
Keywords: Downside risk, fan charts, growth-at-risk, quantile regression.

* Milan Szabo, Czech National Bank and Prague University of Economics and Business, milan.szabo@cnb.cz
I would like to thank Miroslav Plašil, Martin Hodula, Zlatuše Komárková, Tomáš Adam, Katarzyna Budnik, Martin Časta, Karel Brůna and Michal Dvořák. The views expressed in this paper are those of the author and not necessarily those of the Czech National Bank or its management.

1. Introduction

Financial vulnerabilities building up in an economic system increase the uncertainty about future economic growth. Elevated asset prices, a rapid increase in borrowing by businesses and households, excessive risk-taking in the financial sector, and very accommodative monetary policy with its risk-taking channel are examples of vulnerabilities that have been shown to tilt future economic growth to the downside.[1]

These financial vulnerabilities weaken the financial system's ability to withstand negative shocks and unravelling financial imbalances, and so when shocks arrive, losses mount, the financial system weakens, lending stops and economic activity slows by more than it would have done otherwise, potentially leading to an economic downturn or a more severe recession. The focus on downside risks of GDP and the demand to quantify the full and possibly asymmetric GDP distribution rather than point GDP forecasts have increased in recent years.

This paper introduces a new approach to estimating Growth-at-Risk (GaR) employing Bayesian quantile regression. GaR links current financial conditions to the distribution of future GDP growth, focusing on downside risks. GaR can thus be described as a macroeconomic alternative to the various Value-at-Risk models known from risk management in financial institutions.

The proposed Bayesian Growth-at-Risk (BGaR) is unique in making it possible to link the estimated distribution to information obtained from, for example, official GDP projections of central banks. The official GDP projections can then be used as an anchor for the mean, mode or median of the estimated distribution. Alternatively, instead of the official projections, one can link the estimated distribution to information from the results of surveys of professional forecasters, or cast the linkage via other statistics, such as the interquartile range and the standard deviation.

To the same extent, the approach can be applied to obtain linked distribution forecasts of other macroeconomic variables, for example inflation, unemployment or the exchange rate, for which official projections or surveyed forecasts are usually available. Therefore, this unique way of combining the results of quantile regression with projections from different models/information sets can be generalised and is not limited to the presented GaRs. Moreover, the possibility of achieving very close consistency with the official projections of a central bank makes the proposed methodology a promising approach for deriving asymmetric fan charts of macroeconomic prospects – a great improvement over the usual approach of using symmetric confidence bands based on past forecast errors only (see the survey in Franta et al., 2014).

The linkage to official GDP projections introduced in the paper yields many benefits compared to conventional GaRs estimated by known techniques. Firstly, it allows us to control the coherence between the distribution and the official GDP projections. This may be of great importance for central banks, which strive to be credible through transparent and consistent communication (Geraats, 2010). Secondly, the additional information mediated by the linkage may improve the forecasting and the stability of predictions. Furthermore, as the conditional quantiles forming the full distribution are estimated simultaneously, the information from the linkage is propagated among all quantiles – a great improvement over conventional quantile regression algorithms.

[1] See, for example, Borio and Lowe (2002), Schularick and Taylor (2012), Jordà et al. (2013), Adrian and Liang (2014), Mian et al. (2017), López-Salido et al. (2017), Aikman et al. (2018), Adrian et al. (2019) and Aikman et al. (2019).

This may be valuable for countries with short time series. Thirdly, the proposed BGaR prevents quantile-crossing, which is quite usual for conventionally estimated GaRs.

To demonstrate the new approach, the paper presents results for various (B)GaR models estimated using data for the Czech Republic. The results demonstrate the better predictive power of the BGaRs and their more appealing properties for operational use by central banks. Compared to conventionally estimated GaRs, BGaRs achieved 40% improvements in out-of-sample predictions of left tails. The conditional quantiles from BGaRs are stable and do not cross each other. The estimated distributions are consistent with the official GDP projections, giving great communication advantages. The results clearly show the importance of adding the informative linkage and of estimating conditional quantiles simultaneously.

The paper proceeds as follows. After a literature review of GaR models, Section 3 presents the estimation methods, including the new Bayesian approach. Section 4 presents data with model specifications, and Section 5 provides empirical results. Section 6 concludes.

2. Literature Review

Since the work of Cecchetti and Li (2008), who showed the negative impact of booms in equity and housing prices on the quantiles of output using panel–quantile–vector autoregression, the literature focusing on estimation of quantiles in the domain of financial stability has grown substantially. Giglio et al. (2016) showed the great predictive power of a large collection of systemic risk indexes for lower tails of GDP growth using quantile regression with dimension-reduction techniques for advanced economies.

Full-density forecasting was explored by Adrian et al. (2019), who modelled the full distribution of future real GDP growth as a function of current financial and economic conditions. Smoothing the estimated quantiles by fitting a skewed t-distribution, they showed that GDP growth becomes left-skewed during recessions and symmetric during expansions for US data. In Adrian et al. (2018), the authors investigated how downside risks to growth changed over different horizons using panel data for advanced and emerging economies. Following the same definition of growth at risk as the 5% quantile of the predicted growth rate, they showed a positive relationship between financial conditions and tail risk in the near term. These relationships reversed over the medium term: with looser conditions, there was an increase in tail risk at a given horizon.

Adrian et al. (2019) and Giglio et al. (2016) conditioned on financial stress rather than financial vulnerabilities, which may not be very informative for macroprudential surveillance. By contrast, Aikman et al. (2018) aggregated measures of financial vulnerabilities and estimated the distribution of United Kingdom GDP growth conditional on them. With the main focus on the tails, they showed a gradual fall at the fifth quantile prior to the financial crisis. More recently, Aikman et al. (2019) estimated growth at risk for a panel of advanced economies. Employing local projections and panel quantile regression, they found that credit booms, property price booms and wide current account deficits each posed material downside risks to growth at horizons of three to five years.

A more recent paper is Brownlees and Souza (2019), which provides a large study of GaR models, including the "joint GaR" of 24 OECD countries. Figueres and Jarociński (2020) replicated Adrian et al. (2019) for EMU countries and confirmed the relationship between vulnerability of GDP growth and financial conditions for that region. Applications of GaR models for evaluating the effectiveness of macroprudential tools can be found in Duprey et al. (2020), Galán Camacho (2020) and Franta and Gambacorta (2020).

The popularity of GaR models has been boosted by wide interest from central banks[2] and the IMF (International Monetary Fund, 2017; Prasad et al., 2019). However, the quest to operationalise GaR is not always successful, as the quote from Alessandri et al. (2019) suggests:

> We find that, although spikes in financial distress are typically followed by economic contractions, using this relationship for out-of-sample forecasting is not trivial. To some extent, the models predict the slowdowns experienced by Italy after 2008, but the forecasts are volatile, their quality varies across indicators and horizons, and the predictions tend to overestimate the likelihood of an upcoming recession.

This paper aims to propose an alternative and more operational approach that provides more stable predictions and does not overestimate the likelihood of a future recession. This is achieved through the estimator's ability to anchor the distribution to officially projected or surveyed GDP growth point forecasts as well as other statistics. The linkage limits overfitting of the model thanks to an expanded information set provided by a unique and very informative prior. This improves the robustness of the results and yields a substantial communication advantage.

3. Estimation Framework

3.1 Conventional Approach

Estimation of (B)GaR models consists of two steps. In the first step, conditional quantiles from a countable set are derived. The estimated quantiles form inputs to the second step, in which the full distribution of future GDP is derived. Two possible approaches for the second step are kernel density estimation (KDE), and fitting a parametric probability distribution, both outlined in Appendix A.1.

The conventional GaR models in the paper are estimated using the well-studied quantile regression. Denote the year-on-year GDP growth rate as y_{t+h}. The τth quantile is the inverse probability distribution function:

$$\mathbb{Q}_\tau (y_{t+h}) = \inf\{y : P(y_{t+h} < y) \geq \tau\} \tag{1}$$

The quantile function can be represented as the solution to an optimisation problem:

$$\mathbb{Q}_\tau (y_{t+h}) = \arg\inf_q E\left[\rho_\tau (y_{t+h} - q)\right] \tag{2}$$

where $\rho_\tau(x) = x(\tau - \mathbb{1}_{x<0})$ is the quantile loss function, also known as the check function. Furthermore, let us assume that the quantiles take the form of affine functions of covariates $\mathbf{x}_t$.

$$\mathbb{Q}_\tau (y_{t+h} \mid \mathbf{x}_t) = \beta_{\tau,0} + \beta_{\tau,1} x_{t,1} + \cdots + \beta_{\tau,J} x_{t,J} = \mathbf{x}_t \boldsymbol{\beta}_\tau \tag{3}$$

Quantile regression is a general technique for estimating families of conditional quantile functions. Unlike for OLS, there is no closed-form solution for the quantile regression coefficients β_τ, which

[2] See the Financial Stability Reviews of Deutsche Bundesbank (2018), Bank of Lithuania (2019), ECB (2019) and Bank of Japan (2018)

are different with τ but can be solved very efficiently by linear programming methods (Koenker and Bassett Jr, 1978).

In the case of multiple explanatory variables, it is possible to use regularisation methods (Koenker and Bassett Jr, 1978), which should reduce overfitting of the model by adding a penalty. The LASSO regularisation technique is described in Appendix A.2.

Importantly, one can spot that the parameters are not estimated simultaneously for each quantile. The estimation of the conditional median is performed independently of the 25% quantile etc. This may lead to "crossing" of quantiles – very often an issue when estimating conditional quantiles with short time series such as macroeconomic data.

The estimation of the conditional quantiles together with the derivation of the full distribution form the estimator. Let the desired properties of the estimator be (i) an ability to link the distribution to the official GDP projections of central banks etc., providing a communication advantage, (ii) flexibility of the estimated distribution (e.g. bi-modality), (iii) a focus on the whole distribution, avoiding quantile crossing. Regarding the desired properties, the presented conventional estimators of GaR models do not satisfy these properties. KDE allows for flexibility, unlike the usually fitted parametric skewed t-distribution, but there is no way of achieving consistency or avoiding crossing of quantiles for the conventional estimator.

In the next part, the Bayesian approach to estimating BGaR is introduced. This approach is superior to the conventional estimators. Firstly, it avoids crossing of quantiles and estimates all the conditional quantiles simultaneously. Secondly, it allows for a rich set of options to link the estimated distribution to information "outside" the scope of the conventional estimators, including an official GDP projection. BGaR can thus be easily combined with KDE to derive the full and flexible distribution of future GDP growth that satisfies the desired properties.

3.2 Growth-at-Risk: Bayesian Estimation

The standard approach to estimating a Bayesian quantile regression results from the parallel between minimisation of the check function and maximisation of an asymmetric Laplace distribution (Yu and Moyeed, 2001). Kozumi and Kobayashi (2011) elaborated on this by expressing the asymmetric Laplace distribution with a mixture of a standard exponential and a standard normal variable, which leads to the convenient Gibbs sampler. However, this approach still estimates the quantiles separately.

Therefore, the algorithm of Feng et al. (2015) is implemented as the underlying algorithm for the BGaR models introduced. This brings many benefits compared to the usual approaches mentioned above. Firstly, the approach allows us to estimate the joint posterior distribution of multiple m quantiles, leading to higher global efficiency for all quantiles of interest. It does so by treating the Bayesian quantile regression as a semi-parametric problem: the parameter of interest is finite dimensional and the likelihood is nonparametrically obtained through the linearly interpolated density as sketched in Appendix A.3. Then the posterior of all the parameters in the vector $B_m = \left(\boldsymbol{\beta}_{\tau(1)}, \boldsymbol{\beta}_{\tau(2)}, ..., \boldsymbol{\beta}_{\tau(m)}\right)$ can be expressed as:

$$p(B_m|Y) \propto L(Y|B_m) \times \pi(B_m) \tag{4}$$

where $p(B_m|Y)$ is the posterior given all the data Y. The prior for the vector of parameters is denoted as $\pi(B_m)$, and finally $L(Y|B_m)$ is the joint likelihood defined in Appendix A.3.

The joint form of likelihood is crucial in order to establish the linkage between the estimated distribution and the official GDP projection. This is achieved by the "system priors" formulated in Andrle and Benes (2013) and Andrle and Plašil (2018). Simply put, a system prior is an additional prior controlling the behaviour of the model. Since the joint posterior represents the posterior for the entire distribution, the system prior which drives the full distribution can be formulated. For the BGaR models presented in this paper, the system prior is formulated for the estimated median real GDP growths:[3]

$$p(B_m|Y) \propto L(Y|B_m) \times [\pi(B_m) \times \pi_s(\mathbf{X}\boldsymbol{\beta}_{0.5})] \tag{5}$$

The matrix $\mathbf{X}$ is formulated in the next subsection.

3.2.1 Formulation of the Linkage

The main goal of (B)GaR models is to provide predictions. The system priors can be used to enhance the stability of the predictions, provide additional information, and improve coherence with the official GDP projections communicated by the central bank. Furthermore, as the conditional quantiles are estimated simultaneously, the information obtained by the linkage is propagated among the quantiles, including the ones in the left tail.

To achieve that, one can formulate a multivariate normal system prior regularising the predictions of medians for out-of-sample forecasts.

$$\begin{pmatrix} \mathbf{x}_{t-h+1}\boldsymbol{\beta}_{0.5} \\ \dots \\ \mathbf{x}_t\boldsymbol{\beta}_{0.5} \end{pmatrix} \sim N\left[\begin{pmatrix} op_{t,t+1} \\ \dots \\ op_{t,t+h} \end{pmatrix}, \quad \text{diag}(\sigma^2_{op})\right] \tag{6}$$

The system prior formulated above anchors the predicted median GDP growth close to the most recent official GDP projections published at time t ($op_{t,t+h}$ denotes the official GDP projection for time $t+h$ published at time t). The strength of the coherence is governed by the diagonal value σ^2_{op} in the covariance matrix. The presented setting does not set prior beliefs about the covariances.

Alternatively, we can define a system prior linking the behaviour of median GDP growth to the official projections over the entire length of the time series:

$$\begin{pmatrix} \mathbf{x}_1\boldsymbol{\beta}_{0.5} \\ \mathbf{x}_2\boldsymbol{\beta}_{0.5} \\ \dots \\ \mathbf{x}_{t-h}\boldsymbol{\beta}_{0.5} \\ \mathbf{x}_{t-h+1}\boldsymbol{\beta}_{0.5} \\ \dots \\ \mathbf{x}_t\boldsymbol{\beta}_{0.5} \end{pmatrix} \sim N\left[\begin{pmatrix} op_{1,1+h} \\ op_{2,2+h} \\ \dots \\ op_{t-h,t} \\ op_{t-h+1,t+h-1} \\ \dots \\ op_{t,t+h} \end{pmatrix}, \text{diag}(\sigma^2_{op})\right] \tag{7}$$

Such a system prior can be motivated by an effort to reduce overfitting of the model through increased bias in the form of strict backward anchoring to the official projections. The historical

[3] In theory, the system priors can be either substituted or combined with usual priors. The latter approach is followed in the paper.

official GDP projections may not be completely accurate in hindsight but obviously reflect(ed) the "best guess" for future GDP developments based on the information known at the time of publication of the official forecast. This correction may be especially desired for modelling based on macroeconomic time series, which are usually very short and for which quantile regression often attaches a relatively high probability to events such as the global financial crisis (see Alessandri et al., 2019). However, we understand them as examples of "black swans", i.e. low probability events. This fact can be mediated by anchoring the estimated medians to the official GDP predictions along the entire length of the time series, which indirectly informs the model about the extremity of such events. This is a close counterpart of the "bias-variance trade-off".

Both specifications of the system prior will be estimated in the paper. However, we should briefly mention that the system prior offers plenty of ways of controlling the behaviour of the model. Anchoring to the official projection can be chosen by regularising the first moment or the mode of the fitted distribution. Nonetheless, this can lead to much greater computational complexity due to the need to perform a non-parametric estimation (KDE) of the distribution or to fit a parametric probability distribution in each simulation of the Metropolis-Hastings algorithm outlined in Appendix A.3.[4] Alternative forms of linkages, for example to the results of surveys of professional forecasters, are also possible, including other statistics such as the interquartile range.

Besides the system priors $\pi_s(\cdot)$, one can naturally formulate $\pi(B_m)$. For "plain" BGaR we will always assume very loose multivariate normal priors $\pi(B_m)$. On the other hand, the LASSO-BGaR is estimated assuming a multivariate normal prior shrinking some parameters to zero. The setup is discussed in Appendix A.4.

3.3 Evaluation Metrics

Several evaluation metrics are calculated to compare the estimated GaR and BGaR models. The empirical coverage for the given model should be close to $1-\tau$ and is calculated as:

$$EC_\tau = \frac{1}{T}\sum_{t=1}^{T} \mathbb{1}_{[y_{t+h} > \mathbb{Q}_\tau(y_{t+h}|\mathbf{x}_t)]} \tag{8}$$

Gneiting and Raftery (2007) showed that the quantile loss function is a proper metric to compare conditional quantiles. The average quantile loss for a given quantile τ is then stated as:

$$LF_\tau = \frac{1}{T}\sum_{t=1}^{T} \rho_\tau(y_{t+h} - \mathbb{Q}_\tau(y_{t+h} \mid \mathbf{x}_t)) \tag{9}$$

The correctness of the models can also be tested using the DQ test of Engle and Manganelli (2004). Let us define the "hit variable" $H_t = \mathbb{1}_{y_t < \mathbb{Q}_\tau(y_{t+h}|\mathbf{x}_t)}$ for given τ. Sequence H_t should be an independent identical random variable from a Bernoulli distribution with a hit probability equal to τ. The DQ test is based on regression of matrix X built from lagged H_{t-k}. The test statistic is for the null hypothesis, representing an optimal model, stated as $H_0 : \beta_{LS} = 0$:

$$DQ = \frac{\beta_{LS} X' X \beta_{LS}}{p(1-p)} \sim \chi^2_q \tag{10}$$

[4] Regularisation of the first moment obtained from KDE is not very computationally demanding compared to the presented median system prior due to the general validity for KDE: $\int_{-\infty}^{\infty} x\hat{f}_K(x)dx = \frac{1}{M}\sum_{m=1}^{M} \mathbb{Q}_{\tau(m)}(y_{t+h} \mid \mathbf{x}_t)$.

with q equal to the number of columns of the matrix X. χ_q is a Chi-squared distribution of q degrees of freedom.

The median absolute error is also provided for the presented models:

$$MEAE = \frac{1}{T}\sum_{t=1}^{T} |y_{t+h} - \mathbb{Q}_{0.5}(y_{t+h} \mid \mathbf{x}_t)| \tag{11}$$

In addition to the common metrics presented above, coherence with the official GDP projections will be evaluated. Firstly, the average absolute distance between the official projections and the conditional medians is calculated.

$$OPAE = \frac{1}{T}\sum_{t=1}^{T} |op_{t,t+h} - \mathbb{Q}_{0.5}(y_{t+h} \mid \mathbf{x}_t)| \tag{12}$$

Secondly, the average distance between the median (0.5) and the inverse cumulative distribution function (CDF, denoted as $F^{-1}(\cdot)$) at the official projection is calculated.

$$\hat{\tau}_{op} = \frac{1}{T}\sum_{t=1}^{T} |0.5 - F^{-1}(op_{t,t+h})| \tag{13}$$

The lower *OPAE* and $\hat{\tau}_{op}$ are, the more consistent the conditional medians are with the official GDP projections.

4. Data and Model Specification

The estimated GaR and BGaR models are specified with the aim of capturing the measures of financial conditions, vulnerabilities and stress which the Czech National Bank monitors in the pursuit of its financial stability objective. The set of variables includes the following: the financial cycle indicator (FCI), the banking prudence indicator (BPI), current real GDP growth, the financial stress index (CISS) and the global uncertainty index (GEPU), all introduced in Appendix B. The variables enter as the most recently reported, not as their first vintage, even for the out-of-sample predictions introduced in the next section. We are thus concerned mainly with accurately predicting the true macroeconomic state, and arguably the most recent vintage should accurately measure the true state.

Another important input for the BGaR estimate is the official projection published quarterly by the Czech National Bank (CNB) in its Inflation Reports and available on its website. The semi-structural quarterly projection model (QPM) of Beneš et al. (2002) was used until 2008. Subsequently, the CNB switched to a fully structural DSGE model (Andrle et al., 2009; CNB, 2019). Figure A1 shows the decreasing accuracy of the official predictions with increasing prediction horizon (the titles of the panels).

The ultimate specification for the estimated models is:

$$\mathbb{Q}_\tau(y_{t+h} \mid \mathbf{x}_t) = \beta_{\tau,0} + \beta_{\tau,1}y_t + \beta_{\tau,2}FCI_t + \beta_{\tau,3}BPI_t + \beta_{\tau,4}CISS_t + \beta_{\tau,5}GEPU_t \tag{14}$$

The estimated models are the conventional *GaR* and *LASSO-GaR* from Subsection 3.1 and $BGaR^l$, $BGaR^s$, $LASSO\text{-}BGaR^l$ and $LASSO\text{-}BGaR^s$ described in Subsection 3.2. The upper index for the BGaRs denotes the formulation of the system priors. The upper index l stands for the system prior regularising along the whole time series – in-sample and out of sample. The upper index s is the version of the system prior that regularises the out-of-sample forecasts only. Both formulations of the system priors were introduced in Subsection 3.2.1. To save some space, LASSO is abbreviated by l in the tables.

The models are estimated for the quarterly data between 2004Q1 and 2019Q4 and for the following sequence of quantiles τ: $\{\tau(m)\}_{m=1}^{19} = 0.05j$. The models are estimated only for the prediction horizons $h = 1,2,3,4$. The BGaR models are estimated simulating 3.5 million draws with a burn-in of 1.5 million. To anchor the medians, we arbitrarily and very tightly set $\sigma^2_{op} = 10^{-5}$. Parts of the code are implemented in the C++ programming language in order to reduce the computational burden of the BGaR estimator. The conditional quantiles are evaluated with respect to the metrics introduced in Subsection 3.3 as they are, unless stated otherwise, without derivation of the full distribution using KDE or the skewed t-distribution.

5. Empirical Results

Before we delve into the out-of-sample results, we offer a discussion of the in-sample estimates. The text will also illustrate that the best model according to the in-sample results may not be optimal for the primary use of (B)GaR models: predictions.

Table 1 shows some dominance of conventionally estimated GaR models for in-sample estimation of downside risks (set as the 5% quantile). The BGaRs' performance weakens with longer prediction horizons and is naturally lowest for the stricter long-form system prior (the upper index l). The values for the empirical coverage reveal another consequence of linking the full distribution to the official projections. Regularisation and implied constraints on the parameters lead to overestimation of the 5% quantile for $h = 4$. This suggests some degree of optimism in the official GDP forecasts, shifting the next-year distribution to the right. The evolution of the loss function along all the estimated quantiles (Figure A2) indicates that the BGaRs' largest in-sample loss is for the middle quantiles. Obviously, this is primarily due to the linkage. The in-sample consistency with the official GDP projections is, on the other hand, naturally improved for $BGaR^l$, as only this system prior links the full conditional distribution to the GDP projection in-sample (Table 1). The DQ test results are available in the Appendix (Table A1).

Figure 1 shows the detailed evolution of the estimated conditional quantiles derived from the conventional GaR. They are unstable and overestimate the likelihood of an upcoming recession.[5] The conditional quantiles cross each other[6] and the consistency with the official GDP projection is not great. For the out-of-sample starting from 2019Q1, the official GDP projection is close to the right tail (the 95% quantile). In the case of communication of GaR predictions, this could undermine the credibility of the CNB's forecasting apparatus, as the official GDP forecast, which communicates the most probable future path, has a very low probability of occurrence.

[5] Such a decrease may seem like a great predictive ability of the model in light of the ongoing coronavirus pandemic. However, it should be noted that GaR models the risk of GDP growth conditional on the level of loosening of financial conditions, not the tail risk caused by the lockdown of an economy. Above all, the available time series does not cover the period of anti-pandemic measures and the subsequent tensions in financial markets.
[6] Crossing of quantiles is much more common with a finer view according to the τs. For clarity, however, a limited number of quantiles are shown.

Table 1: In-sample Results

	Empirical Coverage, $\tau = 0.05$				Loss Function, $\tau = 0.05$				Median Absolute Error			
	h = 1	h = 2	h = 3	h = 4	h = 1	h = 2	h = 3	h = 4	h = 1	h = 2	h = 3	h = 4
GaR	0.97	0.92	0.93	0.98	0.10	0.14	0.20	0.22	0.88	1.29	1.55	1.77
lGaR	0.97	0.95	0.95	0.95	0.12	0.18	0.23	0.25	1.07	1.51	1.59	1.82
$BGaR^l$	0.95	0.94	0.97	0.95	0.16	0.20	0.25	0.26	1.48	1.85	2.10	2.24
$BGaR^s$	0.98	0.98	0.89	0.88	0.12	0.16	0.22	0.27	0.97	1.39	2.42	2.43
$lBGaR^l$	0.95	0.94	0.95	0.90	0.14	0.21	0.29	0.30	1.47	1.85	2.09	2.24
$lBGaR^s$	0.95	0.92	0.95	0.88	0.13	0.20	0.29	0.29	1.21	2.26	2.00	2.16

	OPAE				$\hat{\tau}_{op}$			
	h = 1	h = 2	h = 3	h = 4	h = 1	h = 2	h = 3	h = 4
GaR	1.22	1.64	1.69	1.76	0.31	0.29	0.28	0.28
lGaR	1.15	1.10	1.41	1.06	0.26	0.20	0.19	0.17
$BGaR^l$	0.65	0.66	0.62	0.54	0.14	0.12	0.10	0.08
$BGaR^s$	1.35	1.83	2.58	2.11	0.24	0.25	0.27	0.23
$lBGaR^l$	0.65	0.67	0.62	0.54	0.09	0.06	0.06	0.05
$lBGaR^s$	1.08	2.28	0.75	1.11	0.20	0.17	0.05	0.11

Figure 2 shows the detailed evolution of the estimated conditional quantiles derived from the conventional $BGaR^l$. This results in great consistency with the CNB's official GDP projection even along the full time series. The conditional quantiles do not cross each other and the prediction stability is improved significantly.

Figure 1: Paths of the Conditional Quantiles Obtained from GaR, h=4

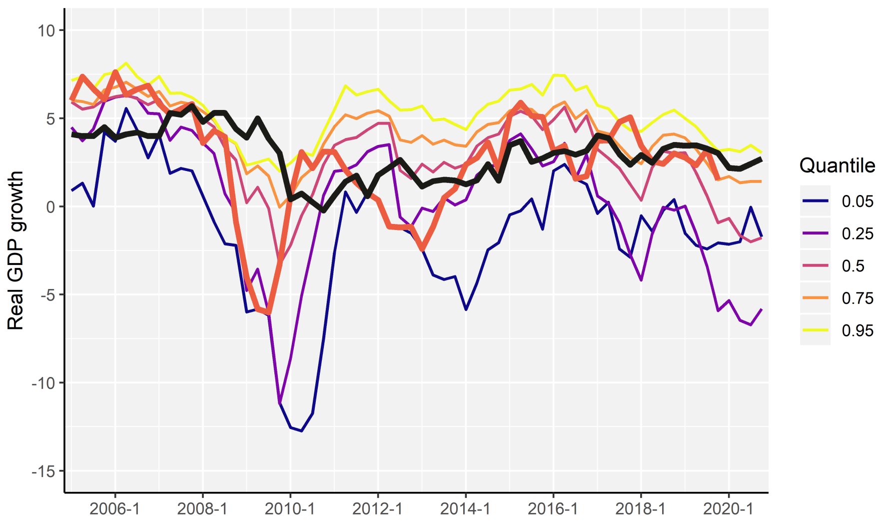


Figure A3 compares the out-of-sample prediction of the distribution of annual real GDP growth for 2020 Q4 obtained using the BGaR model with that published in the official CNB forecast in the Inflation Report issued at the end of 2019. A difference in the flexibility of the modelled distributions can be seen in the figure. The BGaR model makes it possible to reflect the build-up of financial vulnerability in the economy, as the distribution is skewed to the left and the left end is shifted to lower real GDP growth levels (of around -3% to -6%). At the same time, however, it keeps the main mass of the distribution close to the official forecast (around 3%). Unlike the results of the BGaR model, those from the official forecast do not capture the build-up of financial risk at all; the distribution of annual real GDP growth remains symmetric, with its variance governed "solely" by the past forecast errors.

Figure 2: Paths of the Conditional Quantiles Obtained from BGaR^{l}, $h = 4$

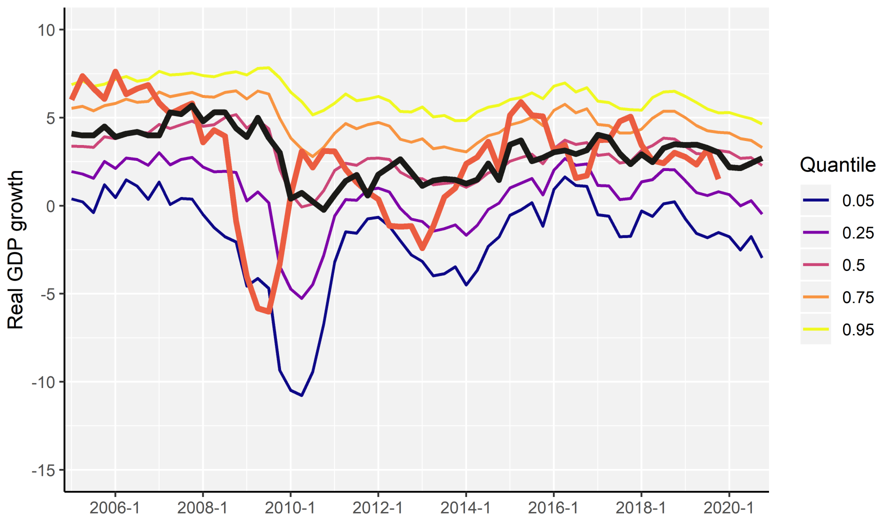


Figure A4 depicts the impact of setting the system prior for the BGaR estimator including confidence bands around the conditional quantiles. The information from the system prior propagates through all the estimated quantiles, changing their location and slightly reducing their uncertainty.

5.1 Pseudo Out-of-sample

To assess and compare the predictive abilities of the models, we repeat the estimation of the models with an increasing number of observations ("pseudo out of sample"). The aim is to evaluate the predictive ability of the model in the conditions in which it would be normally applied. This means that at time T we want to perform a prediction of a conditional quantile $\mathbb{Q}_\tau(y_{T+h} \mid \mathbf{x}_T)$. We therefore estimate the model based on the data until $T-h$ and then perform an out-of-sample prediction for $\mathbb{Q}_\tau(y_{T+h} \mid \mathbf{x}_T)$. The pseudo out-of-sample procedure can be captured by the following algorithm 1:

Algorithm 1: Pseudo out of sample

For time series between T_0 until T and given forecasting horizon h:
 (I) Estimate (B)GaR for time series T_0 until T-h. Get $\mathbb{Q}_\tau(y_{T+h} \mid \mathbf{x}_T)$
 (II) Increase T
Repeat step 1 until y_{T+h} is available

In the case of the BGaR models, the parameters are always estimated using 3.5 million simulations with a burn-in of 1.5 million. This represents $30-h$ iterations of Algorithm 1 for each of the BGaR models. The starting point for the pseudo out-of-sample predictions was set to $T = 2012Q3$.[7] At the annual prediction horizon $h = 4$, there are a total of 494 million simulations. The evaluation of models on pseudo-out-of-sample predictions is only performed for the one-year forecast horizon due to the computational demands. At the same time, it should be noted that due to the number of out-of-sample forecasts (26), the results are only indicative. Caution is warranted especially in the interpretation of the p-values of the DQ tests and the empirical coverage. In addition, $\tau = 0.1$ is considered for the DQ tests.

Table 2 offers a comparison of the estimated models according to the set of metrics selected. The empirical coverage measured for the out-of-sample predictions indicates possible underestimation of the 5% quantiles obtained from the BGaR models and GaR models. This means that out of a total of 26 forecasts of real GDP growth at the 5% quantile, none were lower than real GDP growth in the period. The DQ tests for the models usually do not provide information on the non-optimality of (B)GaR models.

The average loss function reveals great differences between the methods given by a 40% improvement for the BGaRs. MEAE also improved significantly thanks to the linkage. The path of the average loss function for each quantile τ is presented in Figure 3 and confirms the dominance of the BGaRs.

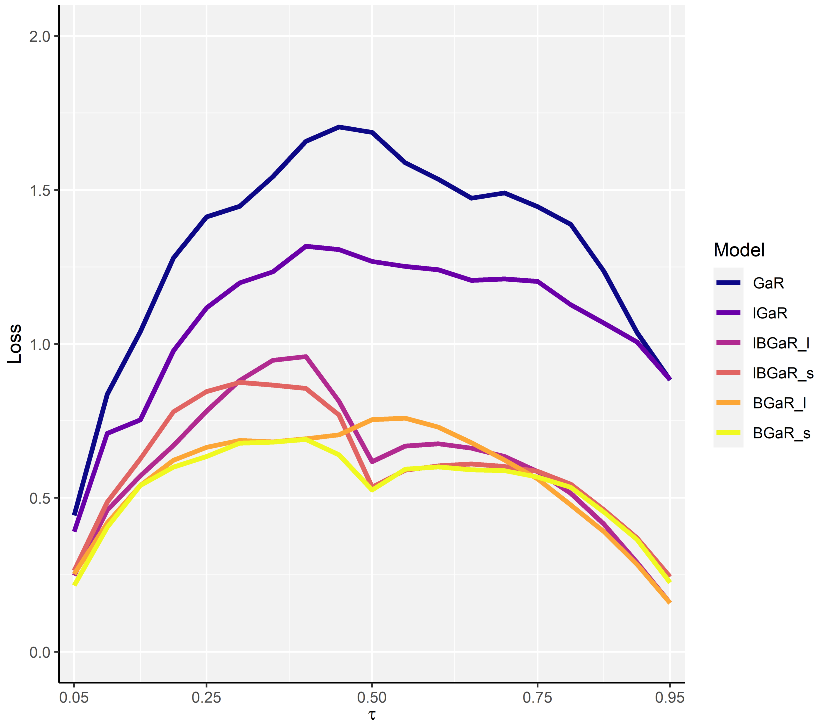
OPAE and $\hat{\tau}_{op}$ – also shown in Table 2 – better demonstrate the consistency of the BGaR models with the official GDP projections compared to the conventionally estimated GaR models. The lower MEAE furthermore correlates with the high coherence between the conditional median and the official projections (see columns OPAE and $\hat{\tau}_{op}$). Unsurprisingly, the greatest consistency for the out-of-sample forecasts is achieved by the BGaR model, which anchors the median to the official projections in the out-of-sample forecasts only. This is due to greater flexibility in matching the main mass of the distribution to the official GDP predictions.

Table 2: Results of Out-of-sample Predictions for h $= 4$

	Empirical Coverage	Loss Function	MEAE	OPAE	$\hat{\tau}_{op}$	DQ test p-value 1lag	2lag	3lag	4lag
GaR	1.00	0.44	3.37	3.31	0.33	0.60	0.75	0.86	0.93
lGaR	1.00	0.39	2.54	2.80	0.34	0.60	0.75	0.86	0.93
$BGaR^l$	0.97	0.25	1.51	0.47	0.06	0.76	0.70	0.82	0.90
$BGaR^s$	0.97	0.26	1.04	0.04	0.02	0.76	0.70	0.82	0.90
$lBGaR^l$	1.00	0.25	1.24	0.50	0.06	0.60	0.75	0.86	0.93
$lBGaR^s$	1.00	0.26	1.07	0.06	0.03	0.62	0.77	0.87	0.94

[7] At the selected origin, the models are estimated with almost 60% of the total length of the time series. This is in line with the generally recommended ratio between training and validation sets. However, one should exercise caution, because the time series is particularly short.

Figure 3: Paths of the Average Loss Functions for the Pseudo Out-of-sample Forecasts



6. Conclusion

Capturing real GDP growth at risk has become very popular. It fits nicely with the empirically proven negative impacts of mounting financial vulnerabilities on future growth. The logic of the approach employed is quite simple: the greater is the vulnerability of the financial system, the more fragile is the estimated future real GDP growth rate and the lower is the level it may reach in a recession. The tail is moving. Central banks need to take tail risk into account in their decisions about the future path of monetary and macroprudential policy. Persistent downside risks to growth can reinforce the need to strengthen the balance sheets of highly indebted firms, households and governments, as well as a banking sector beset by weak profitability and high leverage.

Moreover, estimates for the full GDP growth distribution are useful as a communication tool, because they can aggregate and express the usually dimensionless financial vulnerability or financial stress indices using a macroeconomic variable the public is more familiar with and used to monitoring (GDP growth). Not only that, but the conditional estimate of the distribution of future real GDP growth, and especially the estimate at the 5% quantile, can be used to create an alternative adverse scenario (Arbatli-Saxegaard et al., 2020).

This article showed that the main goal of GaR – predicting – tends to be problematic for the conventional approach to modelling GaR. The predictions tend to overestimate the likelihood of an upcoming crisis, and the quantiles cross each other and are highly unstable in out-of-sample predictions. Communication of the results can also be very tricky. The location of the major mass of the conditional distribution can be very different from the official GDP projection communicated by an official body. The conventional approach is therefore not very operational.

This article proposes a novel approach to modelling the conditional distribution of macroeconomic variables: the Bayesian approach. This approach is unique, as it combines two benefits that improve out-of-sample predictions. Firstly, the parameters for all the quantiles are estimated simultaneously, and the quantiles respect the ordering. Secondly, the approach makes it possible to link the distribution to, for example, official GDP projections. This very informative system prior reduces the risk of overfitting and improves the stability of the predictions and the coherence with the official GDP projections. The results showed a 40% improvement in predicting tail risk compared to the conventional approach and achieved greater consistency with the official GDP projections. BGaRs are more operational and open the way for countries with short time series.

Furthermore, the approach can be employed in different domains, too. The article thus proposed a very general estimator of conditional distributions. The approach can be used to derive asymmetric fan charts for any projected macro-variable or can be employed in combination with surveyed forecasts.

References

ADRIAN, T. AND N. LIANG (2014): "Monetary Policy, Financial Conditions, and Financial Stability." Staff Report No. 690, Federal Reserve Bank of New York.

ADRIAN, T., F. GRINBERG, N. LIANG, AND S. MALIK (2018): "The Term Structure of Growth-at-risk." Working Paper No. 18/180, International Monetary Fund.

ADRIAN, T., N. BOYARCHENKO, AND D. GIANNONE (2019): "Vulnerable Growth." *American Economic Review,* 109(4):1263–89.

AIKMAN, D., J. BRIDGES, S. BURGESS, R. GALLETLY, I. LEVINA, C. O'NEILL, A. VARADI, ET AL. (2018): "Measuring Risks to UK Financial Stability." *Bank of England Staff Working Paper No. 738.*

AIKMAN, D., J. BRIDGES, S. HACIOGLU HOKU, C. O'NEILL, AND R. AKASH (2019): "Credit, Capital and Crises: a GDP-at-Risk Approach." Working Paper No. 824, Bank of England.

ALESSANDRI, P., L. DEL VECCHIO, A. MIGLIETTA, ET AL. (2019): "Financial Conditions and'Growth at Risk'in Italy." Working Paper 1242, Bank of Italy.

ANDRLE, M. AND J. BENES (2013): "System Priors: Formulating Priors about DSGE Models' Properties." Working Paper No. 13/257, International Monetary Fund.

ANDRLE, M. AND M. PLAŠIL (2018): "Econometrics with System Priors." *Economics Letters,* 172:134–137.

ANDRLE, M., T. HLÉDIK, O. KAMENÍK, J. VLCEK, ET AL. (2009): "Implementing the New Structural Model of the Czech National Bank." Working Paper 2/2009, Czech National Bank.

ARBATLI-SAXEGAARD, E. C., K. R. GERDRUP, AND R. M. JOHANSEN (2020): "Financial Imbalances and Medium-term Growth-at-risk in Norway."

AZZALINI, A. AND A. CAPITANIO (2003): "Distributions Generated by Perturbation of Symmetry With Emphasis on a Multivariate Skew T-distribution." *Journal of the Royal Statistical Society: Series B (Statistical Methodology),* 65(2):367–389.

BAKER, S. R., N. BLOOM, AND S. J. DAVIS (2016): "Measuring Economic Policy Uncertainty." *The Quarterly Journal of Economics,* 131(4):1593–1636.

BANK OF JAPAN (2018): *Financial Stability Review.* Bank of Japan.

BANK OF LITHUANIA (2019): *Financial Stability Review.* Bank of Lithuania.

BELLONI, A., V. CHERNOZHUKOV, ET AL. (2011): "ℓ1-penalized Quantile Regression in High-dimensional Sparse Models." *The Annals of Statistics,* 39(1):82–130.

BENEŠ, J., D. VAVRA, J. VLČEK, ET AL. (2002): "Střednědobá Makroekonomická Predikce Makroekonomické Modely v Analytickém Systému ČNB." *Czech Journal of Economics and Finance (Finance a uver),* 52(4):197–231.

BORIO, C. E. AND P. W. LOWE (2002): "Asset Prices, Financial and Monetary Stability: Exploring the Nexus." Working Paper No. 114, Bank for International Settlements.

BROADBENT, B. AND D. G. M. POLICY (2019): "Investment and Uncertainty: the Value of Waiting for News." *Speech given at Imperial College Business School, London, available at https://www. bankofengland. co. uk/speech/2019/benbroadbent-imperial-college-business-school-london.*

BROWNLEES, C. T. AND A. SOUZA (2019): "Backtesting Global Growth-at-Risk." *Available at SSRN: (http:// dx.doi.org/ 10.2139/ ssrn.3461214).*

CECCHETTI, S. G. AND H. LI (2008): "Measuring the Impact of Asset Price Booms Using Quantile Vector Autoregressions." *Brandeis University, Waltham, MA.*

CNB (2019): "Introducing the G3+ Extended Projection Model." Box contained in Inflation Reports III/2019, Czech National Bank.

DEUTSCHE BUNDESBANK (2018): *Financial Stability Review.* Deutsche Bundesbank.

DREHMANN, M., C. E. BORIO, AND K. TSATSARONIS (2012): "Characterising the Financial Cycle: Don't Lose Sight of the Medium Term!" Working Paper No. 380, Bank for International Settlements.

DUPREY, T., A. UEBERFELDT, ET AL. (2020): "Managing GDP Tail Risk." Staff Working Paper 2020-3, Bank of Canada.

ECB (2019): *Financial Stability Review Issue 1.* European Central Bank.

ENGLE, R. F. AND S. MANGANELLI (2004): "CAViaR: Conditional Autoregressive Value at Risk by Regression Quantiles." *Journal of Business & Economic Statistics,* 22(4):367–381.

FENG, Y., Y. CHEN, X. HE, ET AL. (2015): "Bayesian Quantile Regression With Approximate Likelihood." *Bernoulli,* 21(2):832–850.

FIGUERES, J. M. AND M. JAROCIŃSKI (2020): "Vulnerable Growth in the Euro Area: Measuring the Financial Conditions." Working Paper 2458, European Central Bank.

FRANTA, M. AND L. GAMBACORTA (2020): "On the Effects of Macroprudential Policies on Growth-at-risk." *Economics Letters,* 196:1–3.

FRANTA, M., J. BARUNÍK, R. HORVATH, K. SMIDKOVA, ET AL. (2014): "Are Bayesian Fan Charts Useful? The Effect of Zero Lower Bound and Evaluation of Financial Stability Stress Tests." *International Journal of Central Banking,* 10(1):159–188.

GAGLIANONE, W. P. AND L. R. LIMA (2012): "Constructing Density Forecasts From Quantile Regressions." *Journal of Money, Credit and Banking,* 44(8):1589–1607.

GALÁN CAMACHO, J. E. (2020): "The Benefits Are at the Tail: Uncovering the Impact of Macroprudential Policy on Growth-at-risk." Working Paper No. 2007 (2020), Banco de España

GERAATS, P. M. (2010): "Talking Numbers: Central Bank Communications on Monetary Policy and Financial Stability." In *5th ECB Statis tics Conference, available at (http:// www.econ. cam.ac.uk/ people-files/ faculty/ pmg32/ research/ statcom.pdf).*

GIGLIO, S., B. KELLY, AND S. PRUITT (2016): "Systemic Risk and the Macroeconomy: An Empirical Evaluation." *Journal of Financial Economics,* 119(3):457–471.

GNEITING, T. AND A. E. RAFTERY (2007): "Strictly Proper Scoring Rules, Prediction, and Estimation." *Journal of the American statistical Association,* 102(477):359–378.

HOLLO, D., M. KREMER, AND M. LO DUCA (2012): "CISS - A Composite Indicator of Systemic Stress in the Financial System." Working Paper No. 1426, European Central Bank.

INTERNATIONAL MONETARY FUND (2017): *Global Financial Stability Report.* International Monetary Fund.

JORDÀ, Ò., M. SCHULARICK, AND A. M. TAYLOR (2013): "When Credit Bites Back." *Journal of Money, Credit and Banking,* 45(s2):3–28.

KEYNES, J. M. (1936): *The General Theory of Employment, Interest, and Money.* London: Macmillan.

KOENKER, R. AND G. BASSETT JR (1978): "Regression Quantiles." *Econometrica: journal of the Econometric Society,* 46(1):33–50.

KOROBILIS, D. (2017): "Quantile Regression Forecasts of Inflation Under Model Uncertainty." *International Journal of Forecasting,* 33(1):11–20.

KOZUMI, H. AND G. KOBAYASHI (2011): "Gibbs Sampling Methods for Bayesian Quantile Regression." *Journal of Statistical Computation and Simulation,* 81(11):1565–1578.

LÓPEZ-SALIDO, D., J. C. STEIN, AND E. ZAKRAJŠEK (2017): "Credit-market Sentiment and the Business Cycle." *The Quarterly Journal of Economics,* 132(3):1373–1426.

MIAN, A., A. SUFI, AND E. VERNER (2017): "Household Debt and Business Cycles Worldwide." *The Quarterly Journal of Economics,* 132(4):1755–1817.

PFEIFER, L. AND M. HODULA (2018): "A Profit-to-provisioning Approach to Setting the Countercyclical Capital Buffer: the Czech Example." Working Paper 5/2018, Czech National Bank.

PLAŠIL, M., J. SEIDLER, P. HLAVAC, AND T. KONECNY (2014): "An Indicator of the Financial Cycle in the Czech Economy." Thematic article in the Financial Stability Report 2013/2014, Czech National Bank.

PRASAD, M. A., S. ELEKDAG, M. P. JEASAKUL, R. LAFARGUETTE, M. A. ALTER, A. X. FENG, AND C. WANG (2019): "Growth at Risk: Concept and Application in IMF Country Surveillance." Working Paper No. 19/36, International Monetary Fund

SCHULARICK, M. AND A. M. TAYLOR (2012): "Credit Booms Gone Bust: Monetary Policy, Leverage Cycles, and Financial Crises, 1870-2008." *American Economic Review,* 102(2): 1029–61.

SHEATHER, S. J. AND M. C. JONES (1991): "A Reliable Data-based Bandwidth Selection Method for Kernel Density Estimation." *Journal of the Royal Statistical Society: Series B (Methodological),* 53(3):683–690.

YU, K. AND R. A. MOYEED (2001): "Bayesian Quantile Regression." *Statistics & Probability Letters,* 54(4):437–447.

Appendix A: Methodological Appendix

A.1 Deriving the Full Distribution

The KDE is based on a probability density $\hat{f}_K(x)$ of an estimated quantile τ from a sequence $\{\tau(m)\}_{m=1}^{M}$ that can be expressed through a kernel function $k(u)$:

$$\hat{f}_K(x) = \frac{1}{Mh}\sum_{m=1}^{M} k\left(\frac{\mathbb{Q}_{\tau(m)}\left(y_{t+h} \mid \mathbf{x}_t\right) - x}{h}\right) = \frac{1}{Mh}\sum_{m=1}^{M} k\left(u\right) \tag{A1}$$

The estimator $\hat{f}_K(x)$ counts the percentage of conditional quantiles which are close to point x. If many quantiles are near x, then $\hat{f}_K(x)$ is large. Conversely, if only a few quantiles are near x, $\hat{f}_K(x)$ is small. The bandwidth h controls the degree of smoothing (Sheather and Jones, 1991). The usual choice is the second-order Epanechnikov kernel:[8]

$$k(u) = \frac{3}{4}(1-u^2)\mathbb{1}_{|u|\leq 1} \tag{A2}$$

For the fitting of a parametric probability distribution, the skewed t-distribution of Azzalini and Capitanio (2003) is usually chosen (see Adrian et al., 2019). It has density given by:

$$f(y;\mu;\sigma;\alpha;\nu) = \frac{2}{\sigma}t\left(\frac{y-\mu}{\sigma};\nu\right)T\left(\alpha\left(\frac{y-\mu}{\sigma}\right)\sqrt{\frac{\nu+1}{\nu+\frac{y-\mu}{\sigma}}};\nu+1\right) \tag{A3}$$

with $t(\cdot;n)$ and $T(\cdot;n)$ standing for the density and cumulative density function of the standard t-distribution of n degrees of freedom. The skewed t-distribution is then specified with location (μ), scale (σ), shape (α) and degrees of freedom (ν), thus offering great flexibility in terms of shapes.

A.2 LASSO-GaR

The coefficients of the LASSO quantile regression are the result of minimisation according to parameters $\beta_{\tau,j}$, which are elements of the vector $\boldsymbol{\beta}_\tau$:

$$\sum_{t=1}^{T}\rho_\tau\left(y_{t+h} - \mathbb{Q}_\tau\left(y_{t+h} \mid \mathbf{x}_t\right)\right) + \lambda\frac{\sqrt{\tau(1-\tau)}}{T}\sum_{j=1}^{J}\beta_{\tau,j} \tag{A4}$$

Parameter λ is a shrinkage parameter and controls the strength of the penalty. To set the parameter λ one can follow the work of Belloni et al. (2011). Their approach approximates the distribution of the estimation error of the quantile regression parameters by the empirical distribution of a pivotal quantity which accounts for the correlation between predictors. The pivotal quantity is obtained by a total of I simulations of J-dimensional vector $s_{\tau,i} = \sum_{t=1}^{T}\left(\tau - \mathbb{1}_{U_{t,i}-\tau}\right)\mathbf{x}_t$ with $i = 1,2,...,I$ and where $s_{\tau,i}$ is a random draw as $U_{t,b} \sim Unif\,[0;1]$. Hyper-parameter λ is then selected as $\mathbb{Q}_{0.95}\left(\max||s_{\tau,i}||_\infty\right)$. In the next step, the model with significantly non-zero coefficients only is estimated. Then it can have a different set of covariates for given τ. The threshold for the significance of variables in LASSO-type GaR models is set as 10^{-6}.

[8] As in Gaglianone and Lima (2012) and Korobilis (2017).

A.3 BGaR

For the Bayesian approach, the usual starting point is the fact that the posterior distribution of the parameters is proportional to the product of the likelihood and the prior distribution.

$$p(\boldsymbol{\beta}_\tau|Y) \propto L(Y|\boldsymbol{\beta}_\tau) \times \pi(\boldsymbol{\beta}_\tau) \tag{A5}$$

where $p(\boldsymbol{\beta}_\tau|Y)$ is the posterior for the parameter vector introduced above, given all the data Y. The prior for the vector of parameters is denoted as $\pi(\boldsymbol{\beta}_\tau)$, and finally $L(Y|\boldsymbol{\beta}_\tau)$ is the likelihood given the parameters.

The linearly interpolated density for pairs (y_{t+h}, x_t) over $t = 1, 2, ..., T-h$ is:

$$\begin{aligned}\hat{f}_t(y_{t+h} \mid x_t) = &\left(\sum_{j=1}^{m-1} \mathbb{1}_{\left[y_{t+h} \in \left(x_t\beta_{\tau(j)}, x_t\beta_{\tau(j+1)}\right)\right]} \frac{\tau(j+1)-\tau(j)}{x_t\beta_{\tau(j+1)} - x_t\beta_{\tau(j)}}\right) + \\ &+ \mathbb{1}_{\left[y_{t+h} \in \left(-\infty, x_t\beta_{\tau(1)}\right)\right]} \tau(1) f_1(y_{t+h}) + \\ &+ \mathbb{1}_{\left[y_{t+h} \in \left(x_t\beta_{\tau(m),\infty}\right)\right]} (1-\tau(m)) f_2(y_{t+h})\end{aligned} \tag{A6}$$

where $\tau(j)$ is the j-th element of a vector of estimated quantiles $(\tau(1), \tau(2), ..., \tau(m))$ and f_1 is the density of the normal distribution $\mathcal{N}(x_t\beta_{\tau(1)}, \sigma^2)$. Similarly, f_2 is $\mathcal{N}(x_t\beta_{\tau(m)}, \sigma^2)$ (see Feng et al., 2015).

Moreover, let us define vector $B_m = \left(\beta_{\tau(1)}, \beta_{\tau(2)}, ..., \beta_{\tau(m)}\right)$. Then the approximated likelihood $L(Y|\beta_\tau)$ is as follows, where the accuracy of approximation is increasing with m (Feng et al., 2015):

$$L(Y|B_m) \approx \prod_{t=1}^{T-h} \hat{f}_t(y_{t+h} \mid x_t) \tag{A7}$$

The joint-posterior is then:

$$p(B_m|Y) \propto L(Y|B_m) \times \pi(B_m) \tag{A8}$$

The parameters are drawn using a modified Metropolis-Hastings algorithm.

From the steps of Algorithm 2, the high computational complexity of this approach can be observed (see steps 2 and 3 of the algorithm). This is further exacerbated by the need for a sufficiently high number of estimated quantiles to ensure greater accuracy of the linear approximation of the likelihood function. The algorithm then easily leads to a high dimension of the parameter vector B_m, which requires a large number of Metropolis-Hastings simulations. For example, for a model of 5 variables and 19 estimated quantiles, it is necessary to estimate 114 parameters. Theoretically, the sampling of a given parameter occurs in one of the 114 simulations.

Algorithm 2: Modified Metropolis-Hastings

Choose an initial value for B_m^0 , for example using frequentist quantile regression. Order the intercepts $(\beta_{\tau(1),0}, \beta_{\tau(2),0}, ..., \beta_{\tau(m),0})$ from the smallest to the largest.

Calculate $L^0 = \prod_{t=1}^{T-h} \hat{f}^0\left(y_t + h \mid x_t, B_m^0\right)$, or L^{k-1} in the k-th simulation.

Randomly draw m and j determining the index of $\beta_{\tau(m),j}$, which will be sampled according to the algorithm described in Feng et al. (2015). This will give a new B_m^*. The way the parameters are sampled does not allow for quantile crossing.

Calculate L^* for B_m^*.

Calculate the acceptance:

$$\alpha = \min\left(1, \frac{\pi(B_m^*)L^*\pi_s(B_m^*)}{\pi(B_m^{k-1})L^0\pi_s(B_m^{k-1})}\right)$$

Repeat for the selected number of simulations.

A.4 LASSO-BGaR

For LASSO-BGaR, the following multivariate normal prior is imposed:

$$B_m \sim N\left[\begin{pmatrix} 0 \\ 0 \\ ... \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \phi_{\tau(1),0} & 0 & ... & 0 & 0 \\ 0 & \phi_{\tau(1),1} & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & \phi_{\tau(m),J-1} & 0 \\ 0 & 0 & ... & 0 & \phi_{\tau(m),J} \end{pmatrix}\right] \tag{A9}$$

where $\phi_{\tau(m),j}$ for $j = 0, 1, ..., J$ in the covariance matrix denotes a variable with two possible values. Either the parameter $\beta_{\tau(m),j}$ is not significant according to the algorithm of Belloni et al. (2011), then $\phi_{\tau(m),j}$ takes a small value shrinking parameter $\beta_{\tau(m),j}$, or the parameter is significant and $\phi_{\tau(m),J}$ takes a large value leaving the data to fully determine $\beta_{\tau(m),j}$. The hyperparameter $\phi_{\tau(m),j}$ for the formulated *LASSO-BGaR* is set to 1,000 or 10^{-4} with respect to the significance of the given variable. The threshold for the significance of variables in LASSO-type BGaR models is set as 10^{-6}.

Appendix B: Data

The FCI of Plašil et al. (2014) measures systemic risks across the financial cycle by compressing the development of multiple sub-indicators such as excessive growth in real estate prices, loans to the real sector and indebtedness of the real sector. At the same time, the indicator also captures the correlations of the subindicators.

Due to the importance of the banking sector for the Czech economy, the banking prudence indicator (BPI) of Pfeifer and Hodula (2018) is also employed, in the form of the ratio of the total interest margin to loss provisions per unit of private loans (i.e. loans to households and non-financial corporations).

The construction of the CISS indicator is based on the method presented in Hollo et al. (2012). The indicator expresses stress on the Czech financial market through the spreads and volatility of interbank interest rates, interest rate swap rates, government bond yields , stock prices and the exchange rate. The CISS simply reflects the deteriorating ability to mediate market financing and diversify financial risks in the market.

The presence of annual GDP growth over the past year reflects empirical evidence that recessions are deeper when they coincide with contractions in the financial cycle (Drehmann et al., 2012).

The negative effects of increased uncertainty in the economy can be linked to the work of Keynes (1936) and Broadbent and Policy (2019). In order to take into account the negative impact of increased uncertainty in the distribution of future GDP growth, the model was enriched with the Global Economic Policy Uncertainty (GEPU) index (see Baker et al., 2016). Baker et al. (2016) show that innovations to their index precede declines in investment, output and employment for the USA and 12 major economies. The GEPU is a PPP-adjusted GDP-weighted average of national EPU indices for 21 countries.[9] Each national EPU index reflects the relative frequency of own-country newspaper articles that contain a trio of terms pertaining to the economy (E), policy (P) and uncertainty (U).

[9] Australia, Brazil, Canada, Chile, China, Colombia, France, Germany, Greece, India, Ireland, Italy, Japan, Mexico, the Netherlands, Russia, South Korea, Spain, Sweden, the United Kingdom and the United States.

Appendix C: Additional Tables and Figures

Figure A1: CNB Forecasts and Realised GDP Growth Rates

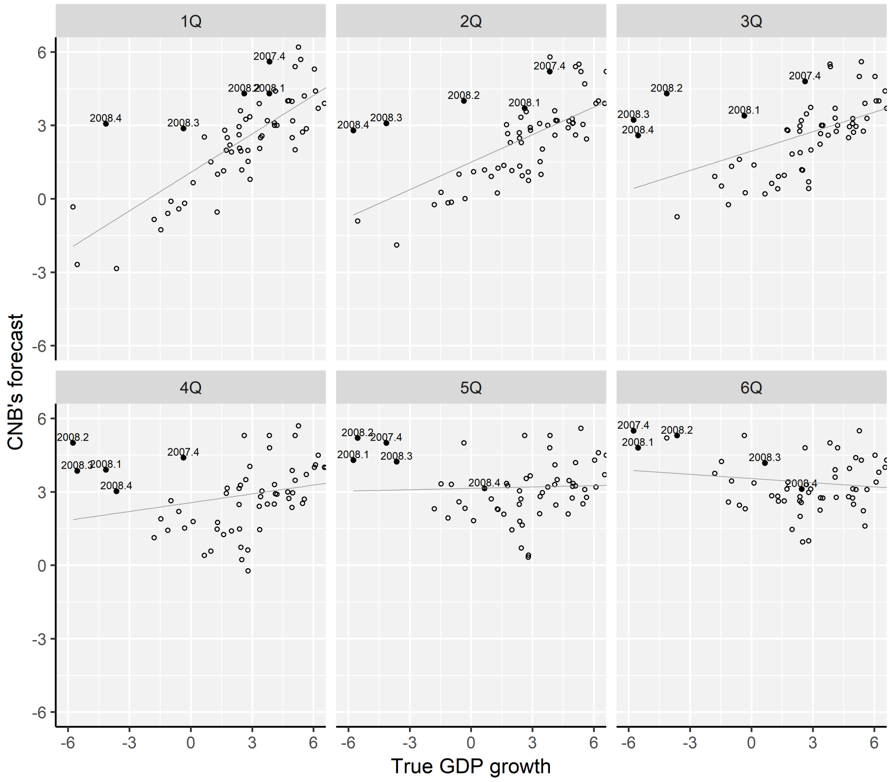


Table A1: In-Sample Results of DQ Tests for $\tau = 0.05$

	DQ test p–value (1lag)				DQ test p–value (2lag)				DQ test p–value (3lag)				DQ test p–value (4lag)			
	h = 1	h = 2	h = 3	h = 4	h = 1	h = 2	h = 3	h = 4	h = 1	h = 2	h = 3	h = 4	h = 1	h = 2	h = 3	h = 4
GaR	**0.76**	**0.13**	**0.14**	**0.69**	**0.87**	**0.12**	**0.09**	**0.81**	**0.93**	**0.19**	**0.15**	**0.89**	**0.97**	**0.22**	**0.15**	**0.94**
lGaR	**0.88**	0.03	**0.72**	0.03	**0.93**	0.02	**0.81**	0.02	**0.96**	0.02	**0.12**	0.02	**0.98**	0.02	**0.18**	0.02
$BGaR^l$	**0.05**	0.00	0.02	**0.24**	**0.09**	0.00	0.01	**0.19**	**0.07**	0.00	0.02	**0.27**	0.03	0.00	0.02	**0.25**
$BGaR^s$	**0.82**	**0.74**	0.00	0.00	**0.91**	**0.85**	0.00	0.00	**0.95**	**0.91**	0.00	0.00	**0.96**	**0.94**	0.01	0.00
$lBGaR^l$	0.04	0.00	0.13	0.00	**0.07**	0.00	0.11	0.00	**0.05**	0.00	**0.16**	0.00	0.02	0.00	**0.13**	0.00
$lBGaR^s$	**0.89**	**0.15**	**0.13**	0.00	**0.91**	**0.16**	**0.11**	0.00	**0.92**	**0.21**	**0.16**	0.00	**0.94**	**0.18**	**0.13**	0.00

Figure A2: Paths of the Average Loss Functions for In-Sample Forecasts.

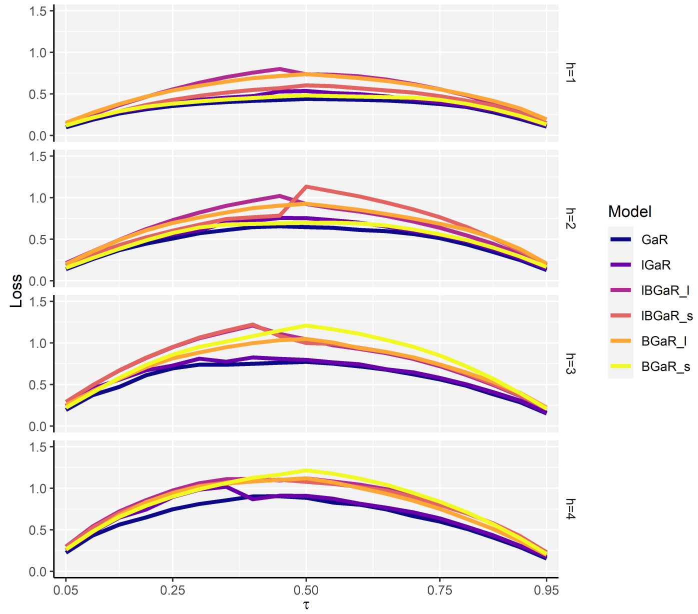
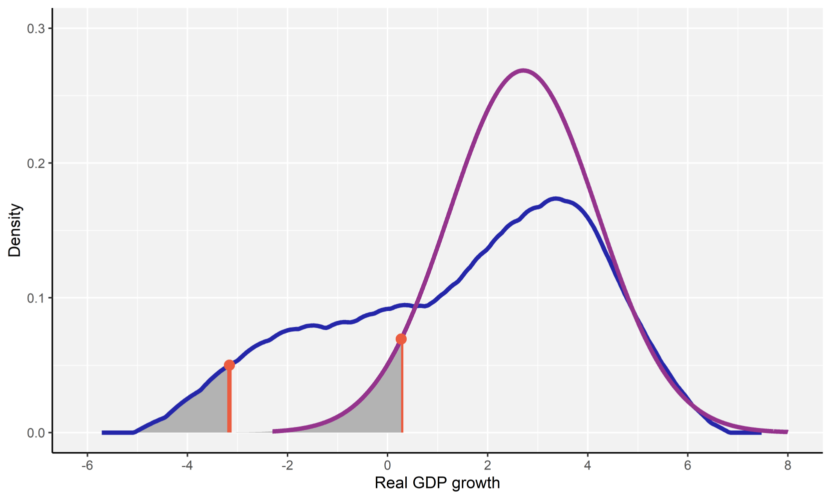
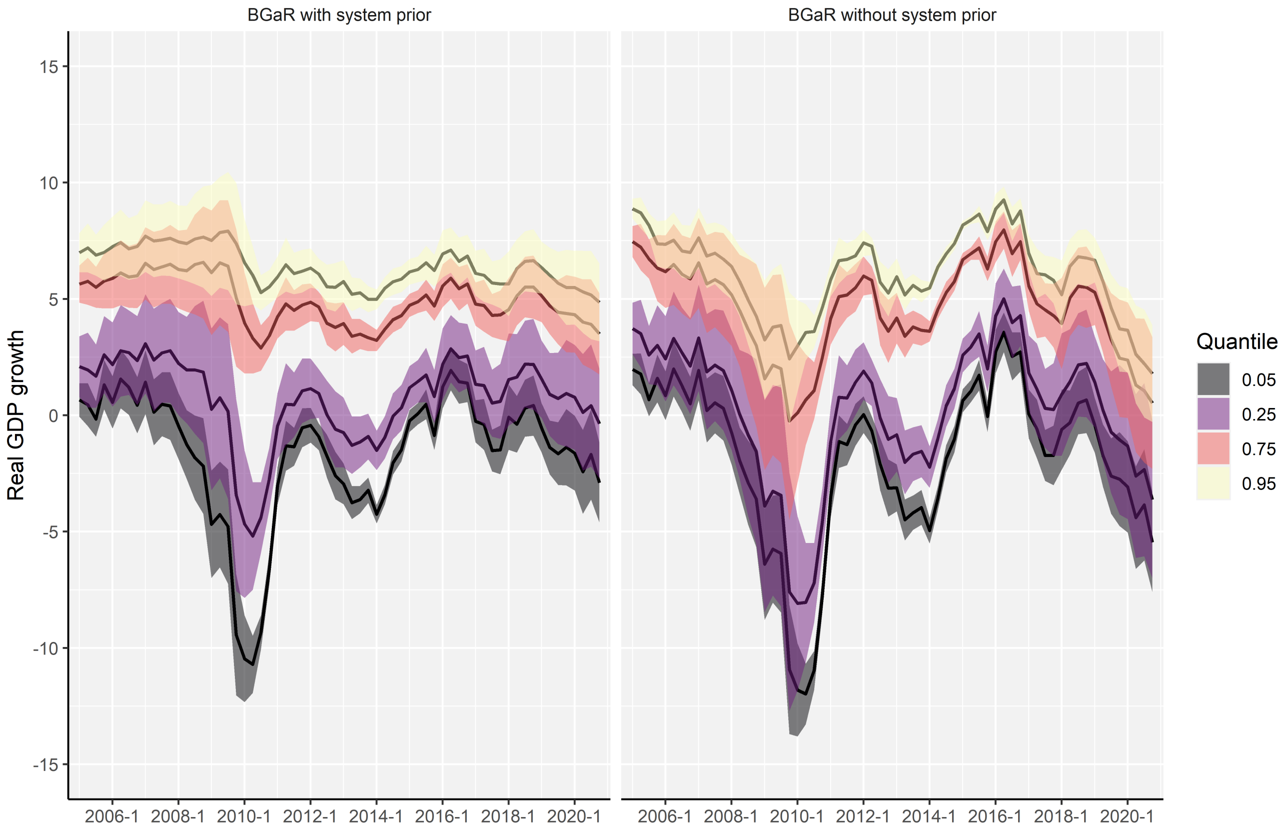


Figure A3: Comparison of Asymmetric and Symmetric Forecasts



Estimated distribution for 2020Q4 based on BGaR in blue, Epanechnikov kernel used. Distribution from 2020Q4 official forecast published in December Inflation Report 2019 in purple. Shaded area is 5% left tail.

Figure A4: Paths of Conditional Quantiles for BGaR



Evolution of conditional quantiles for BGaR with and without system prior. Bands represent confidence derived from ± standard deviation of simulated chains for all parameters.

CNB Working Paper Series (since 2019)

WP 3/2020	Milan Szabo	*Growth-at-risk: Bayesian approach*
WP 2/2020	Martin Hodula Ngoc Anh Ngo	*Finance, growth and (macro)prudential policy: European evidence*
WP 1/2020	Michal Franta Ivan Sutóris	*Dynamics of Czech inflation: The role of the trend and the cycle*
WP 12/2019	Josef Bajzík Tomáš Havránek Zuzana Iršová Jiří Schwarz	*The elasticity of substitution between domestic and foreign goods: A quantitative survey*
WP 11/2019	Martin Hodula Simona Malovaná Jan Frait	*Too much of a good thing? Households' macroeconomic conditions and credit dynamics*
WP 10/2019	Martin Hodula Simona Malovaná Jan Frait	*Introducing a new index of households' macroeconomic conditions*
WP 9/2019	Jiří Panoš Petr Polák	*How to improve the model selection procedure within a stress testing framework?*
WP 8/2019	Sebastian Gechert Tomáš Havránek Zuzana Iršová Dominika Kolcunová	*Death to the Cobb-Douglas production function? A quantitative survey of the capital-labor substitution elasticity*
WP 7/2019	Alexis Derviz	*Coexistence of physical and crypto assets in a stochastic endogenous growth model*
WP 6/2019	Dominika Ehrenbergerová Simona Malovaná	*Introducing macro-financial variables into semi-structural model*
WP 5/2019	Martin Hodula	*Monetary policy and shadow banking: Trapped between a rock and a hard place*
WP 4/2019	Simona Malovaná Žaneta Tesařová	*Banks' credit losses and provisioning over the business cycle: Implications for IFRS 9*
WP 3/2019	Aleš Bulíř Jan Vlček	*Monetary policy is not always systematic and data-driven: Evidence from the yield curve*
WP 2/2019	Dominika Kolcunová Simona Malovaná	*The effect of higher capital requirements on bank lending: The capital surplus matters*
WP 1/2019	Jaromír Baxa Tomáš Šestořád	*The Czech exchange rate floor: Depreciation without inflation?*

CNB Research and Policy Notes (since 2019)

RPN 1/2020	Simona Malovaná	*How to organize research in central banks: The Czech National Bank's experience*
RPN 4/2019	Tomáš Konečný Lukáš Pfeifer	*Macroprudential ring-fencing*
RPN 3/2019	Petr Polák Jiří Panoš	*Impact of expectations on IFRS 9 loan loss provisions*
RPN 2/2019	Jan Filáček Ivan Sutóris	*Inflation targeting flexibility: The CNB's reaction function under scrutiny*
RPN 1/2019	Iveta Polášková Luboš Komárek Michal Škoda	*The contemporary role of gold in central banks' balance sheets*

CZECH NATIONAL BANK
Na Příkopě 28
115 03 Praha 1
Czech Republic

ECONOMIC RESEARCH DIVISION
Tel.: +420 224 412 321
Fax: +420 224 412 329
http://www.cnb.cz
e-mail: research@cnb.cz

ISSN 1803-7070