WORKING PAPER SERIES

Sparse Restricted Perception Equilibrium

Volha Audzei
Sergey Slobodyan

8/2018

Sparse Restricted Perception Equilibrium

Volha Audzei and Sergey Slobodyan *

Abstract

In this paper we study model selection under bounded rationality and the impact of monetary policy on the equilibrium choice of forecasting models. We use the concept of sparse rationality (developed recently by Gabaix, 2014), where paying attention to all possible variables is costly and agents can choose to over- or under-emphasize particular variables, even fully excluding some of them. Our main question is whether an initially mis-specified equilibrium (the restricted perceptions equilibrium, or RPE) is compatible with the equilibrium choice of sparse weights describing the allocation of attention to different variables by the agents inhabiting this RPE. In a simple New Keynesian model, we find that the agents stick to their initial mis-specified AR(1) forecasting model choice when monetary policy is less aggressive or inflation is more persistent. We also identify a region in the parameter space where the agents find it advantageous to pay attention to no variable at all.

Abstrakt

V této práci zkoumáme výběr modelu v rámci omezené racionality a dopad měnové politiky na rovnovážnou volbu prognostických modelů. Používáme koncept řídké racionality (který v nedávné době vypracoval Gabaix, 2014), v němž je nákladné věnovat pozornost všem možným proměnným a ekonomické subjekty mohou svojí volbou nadhodnocovat či podhodnocovat konkrétní proměnné, nebo dokonce některé z nich zcela vynechat. Hlavní otázkou je, zda zprvu chybně specifikovaná rovnováha (Restricted Perceptions Equilibrium, RPE) je v souladu s rovnovážnou volbou řídkých vah, které popisují přiřazení pozornosti různým proměnným subjekty v rámci RPE. V jednoduchém novokeynesiánském modelu docházíme k závěru, že subjekty se drží svého prvotního chybně specifikovaného prognostického modelu AR(1), je-li měnová politika méně agresivní nebo se inflace stane perzistentnější. V parametrickém prostoru rovněž nacházíme oblast, kde subjekty považují za výhodné nevěnovat pozornost žádné proměnné.

JEL Codes: D84, E31, E37.
Keywords: Bounded rationality, expectations, learning, model selection.

---

*VolhaAudzei, Czech National Bank, Economic Research Division, Monetary Department, and CERGE-EI, a jointworkplace of Charles University and the Economics Institute of the Czech Academy of Sciences, Prague, CzechRepublic, vaudzei@cerge-ei.cz.
Sergey Slobodyan, CERGE-EI, a joint workplace of Charles University and the Economics Institute of the Czech Academy of Sciences, Prague, Czech Republic, and National Research University Higher School of Economics, St. Petersburg, Russia, sergey.slobodyan@cerge-ei.cz.
We thank Filip Matejka, Xavier Gabaix, Raf Wouters, Pierpaolo Benigno, Alexis Derviz, and participants of various conferences and workshops. The work has benefitedfromthefinancialsupportby theCzechScienceFoundation under Project No. P402/12/G097 DYME, Dynamic Models in Economics and by the European Unions Seventh Framework Program (FP7) for research, technological development and demonstration under grant agreement number 612796. The views expressed in this paper are those of the authors and not necessarily those of the Czech National Bank.

Nontechnical Summary

There is a growing literature showing that economic agents form their expectations in a way that is often inconsistent with the rational expectations hypothesis. Agents' expectations, in turn, influence the efficiency of monetary policy and may even call for a discussion of different policy choices (e.g. Gabaix 2016, García-Schmidt and Woodford 2015, and Hommes et al. 2017).

In this paper we contribute to the literature by studying how monetary policy itself influences the way agents form their expectations. We consider a stylized model where adaptively learning agents are restricted to the use of only a subset of the observables. This subset is chosen such that it results in the smallest forecast errors. We then allow our agents to reconsider their initial choice of forecasting rules subject to information constraints as in Gabaix (2014). We then study under which conditions agents stick to the initially mis-specified forecasting rule, move to another mis-specified rule, or switch a rule consistent with rational expectations.

Within a textbook New Keynesian model framework, we show that one of the crucial parameters governing the agents' choice is the reaction of monetary policy to inflation expectations in the Taylor rule. The stronger is the reaction of monetary policy to the deviation of inflation expectations from the target, the larger is the parameter space where a rational expectations-consistent rule is selected. We consider two basic policy rules, one with forward-looking inflation expectations and the other with contemporaneous inflation expectations. The predictions under the two rules are similar, except that the rule with contemporaneous expectations results in lower volatility, and less tightening is needed for the agents to move to the rational expectations equilibrium.

We also find that as inflation becomes more persistent, the mis-specified AR(1) rule for inflation forecasting survives in a larger parameter region. The same prediction holds for a smaller correlation between inflation and output. These results are supported by some studies on professional forecasters' behavior. An example is Lopez-Perez (2017), where forecasters are found to have paid more attention to inflation and less to the output gap in the recent years of high inflation persistence and low correlation between inflation and the output gap.

1. Introduction

It has been understood for a long time that the hypothesis of rational expectations (RE), while delivering a theoretically elegant, model-consistent, and typically unique solution for agents' expectations, imposes cognitive and computational demands on them that might be incompatible with reality. As a result, deviations from RE have been studied in a growing stream of theoretical and empirical literature, including the bounded rationality (Marcet and Sargent 1989), adaptive learning (Evans and Honkapohja 2001), sticky information (Mankiw and Reis 2007), rational inattention (Sims 2003), and sparse rationality (Gabaix 2014) approaches.

The question of model selection under model uncertainty has been on econometricians' minds for decades. Recently, with increasing computer power and the development of machine learning applications, interest in this question has been revived. Computer routines have been developed to select models with greater explanatory power or to compute weights to combine a vast number of models. Because including more variables always improves the model fit to the data but increases the variance of the estimators, it is important to control for "overfitting", that is, to select the minimum set of variables necessary to produce the efficient estimator. One tool for reducing the number of regressors while penalizing for overfitting is the Lasso estimator, originally developed by Tibshirani (1996). In the context of macroeconomics, the studies that use the Lasso estimator for regression shrinkage include Zou and Hastie (2005) and Gefang (2014). This idea of sparsity in selecting the important variables has been adopted by Gabaix (2014) within his concept of sparse rationality, where instead of regression coefficients agents estimate the attention weights. The penalty term in the Lasso objective function then gets a straightforward interpretation as the attention cost. In this paper, we use the sparce rationality concept to study the persistence of and switching between the mis-specified forecasting rules. We address rule selection in a small theoretical model, though the insights can be extended to larger models when our individual variables are interpreted as groups of potential regressors.

Our paper is related to the literature on adaptive learning (AL) and bounded rationality. In the adaptive-learning approach to modeling deviations from the rational expectations (RE), agents are assumed to possess little prior knowledge of the underlying structure of the economy, and to gradually learn the coefficients in their forecasting rules using econometric methods. A survey of this approach to learning in macroeconomics can be found in Evans and Honkapohja (2009). Several papers have shown that adaptive agents can persist in using forecasting rules that are mis-specified relative to RE ones. Molnar (2007) models a class of agents who learn what the best forecasting rule is given the past data. Even if their forecasting rules are mis-specified, such learners can survive competition with RE agents. In Evans et al. (2012), convergence to a mis-specified equilibrium happens when the expectations feedback is strong. Adam (2005) considers an economy where agents are restricted to processing only a certain number of variables in the regression and thus to using underparametrized forecasting rules. As the agents' expectations affect the data-generating process of the model and induce a restricted perceptions equilibrium (RPE), the restricted rule can outperform the rational expectations rule in equilibrium. Similarly to Evans et al. (2012), this happens when there is large enough feedback from expectations to the outcome variable.

Another way to justify agents' use of mis-specified forecasting rules is to assume that they have limited information-processing capacity, as is done in the rational inattention literature — see Sims (2003), Mackowiak and Wiederholt (2009), and Matejka and McKay (2015). In this literature, attention allocation is based on the concept of enthropy. Sparce rationality of Gabaix (2014) is a different, less computationally demanding, approach to the attention allocation problem, with agents assigning attention weights to variables based on their relative informational content.

Our paper contributes to the literature by making connections between two related but distinct concepts, namely, adaptive learning and sparse rationality. In a simple New Keynesian model of monetary policy, the adaptively learning agents choose a strict subset of variables from the RE equilibrium set for their forecasting functions, thus inducing a restricted perceptions equilibrium (RPE). We then allow agents inhabiting this RPE to reconsider their forecast rules, imposing the informational cost constraint modeled as in Gabaix (2014). We ask whether the stability of the RPE is sufficient to ensure that the same subset of variables is selected by informationally constrained agents. In other words, we are interested in whether the initial mis-specification becomes self-perpetuating in the case of informational constraints. We find that one of the key parameters favoring the survival of a mis-specified rule is the strength of the expectational feedback, which is a function of monetary policy aggressiveness. We show that more aggressive monetary policy increases the region in the parameter space where the agents move to the RE equilibrium. We consider two policy rules, one with forward-looking inflation expectations and the other with contemporaneous inflation expectations. Under the policy rule with contemporaneous inflation expectations, the region in the parameter space where a mis-specified rule survives in equilibrium is smaller.

There are now a number of papers that show how bounded rationality influences monetary and fiscal policy transmission. Among them are Gabaix (2016) and García-Schmidt and Woodford (2015). Some of their findings call for reconsideration of the Taylor principle and explain the forward guidance puzzle. As agents do not see far into the future in Gabaix (2016), current policy has a limited effect on their future decisions. Likewise, the announcement of future policies has an attenuated effect on agents' contemporaneous decisions. We, however, are interested in how monetary policy influences expectation formation, rather than in how expectation formation influences monetary policy.

Another strand of research on bounded rationality we relate to is experimental research on the interaction of monetary policy and agents' expectations. Examples include Pfajfar and Žakelj (2018) and Assenza et al. (2013), who find that monetary policy aggressiveness influences the choice of forecasting rule. Studies by Hommes (2014) and Heemeijer et al. (2009) support the importance of the expectational feedback parameter for the survival of mis-specified rules. In our paper, the expectational feedback parameter is the inverse of monetary policy aggressiveness, and we study how it influences the agents' modeling choice.

The paper is organized as follows. In the second section we study an economy where agents learn about a simple exogenous process. We derive analytical results and provide economic intuition for them. In the third section we move to a simple New Keynesian model and study the interaction of monetary policy and agents' forecasting rules. The last section concludes.

2. Simple Model

In this section we demonstrate the main intuition behind our results in a simple model. Later in the paper, we generalize our findings to a three-equation New Keynesian model and discuss the possible policy implications. We start our analysis with a simple process:

$$y_t = \alpha + \beta E_t y_{t+1} + \gamma_1 w_t^1 + \gamma_2 w_t^2 + \eta_t, \tag{1}$$

where w_t^1 and w_t^2 are persistent observable shocks such that

$$\begin{bmatrix} w_t^1 \\ w_t^2 \end{bmatrix} = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} \begin{bmatrix} w_{t-1}^1 \\ w_{t-1}^2 \end{bmatrix} + \begin{bmatrix} \varepsilon_t^1 \\ \varepsilon_t^2 \end{bmatrix}.$$

The $\left(w_t^1, w_t^2\right)$ shocks are normally distributed around zero with variance-covariance matrix

$$\Sigma^w = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix},$$

with $\rho \in [-1,1]$ being the correlation coefficient between the shocks, defined as $\rho = \frac{Cov\left(w_t^1, w_t^2\right)}{\sigma_1\sigma_2}$. η_t is an *iid*, normally distributed shock around zero mean with variance σ_η. Without loss of generality, γ_1 and γ_2 could be taken to be positive.

To address the model selection question using a shrinkage estimator with a fixed penalty term, it is necessary to standardize the variables, that is, to subtract their mean and divide by their standard deviation. Note that when the variables are standardized, the coefficients are adjusted appropriately. To save on notation, in what follows we refer to variables in their standardized form unless stated otherwise: $y_t = \frac{y_t - \bar{y}}{\sigma_y} = \frac{y_t}{\sigma_y}$, $w_t^i = \frac{w_t^i - \bar{w}^i}{\sigma_i} = \frac{w_t^i}{\sigma_i}$, $Cov(w_t^1, w_t^2) = \frac{Cov(w_t^1, w_t^2)}{\sigma_1\sigma_2} = \frac{\rho\sigma_1\sigma_2}{\sigma_1\sigma_2} = \rho$. The variances of the standardized variables are equal to one. As for the coefficients, their transformed values are given by $\beta = \beta\frac{\sigma_y}{\sigma_y} = \beta$, $\gamma_i = \gamma_i\frac{\sigma_i}{\sigma_y}$.

The RE minimum state variable (MSV) solution of this model is given by

$$y_t = a + g_1 w_t^1 + g_2 w_t^2 + \eta_t, \tag{2}$$

with MSV coefficients:

$$g_1 = \frac{\gamma_1}{1 - \beta\rho_1}, \tag{3}$$
$$g_2 = \frac{\gamma_2}{1 - \beta\rho_2}. \tag{4}$$

We restrict the agents to using only one variable in their forecasting models, as in Adam (2005, 2007) framework.[1] In our model, their forecasting rule could use either w_t^1 or w_t^2:

$$y_t = a_1 + b_1 w_t^1 + \eta_t, \tag{5}$$
$$y_t = a_2 + b_2 w_t^2 + \eta_t. \tag{6}$$

Without loss of generality, we assume that our agents use (5) as their perceived law of motion (PLM), which, when the agents use this PLM to form expectations about y_{t+1}, induces the restricted perceptions equilibrium that we call RPE1. Substituting the forecast formed using (5) into (1), we obtain the actual law of motion (ALM):

$$y_t = \alpha + \beta a_1 + \bar{b}_1 w_t^1 + \bar{b}_2 w_t^2 + \eta_t, \tag{7}$$

with

$$\bar{b}_1 = \beta\rho_1 b_1 + \gamma_1, \tag{8}$$
$$\bar{b}_2 = \gamma_2. \tag{9}$$

[1] Such a restriction is motivated by empirical and experimental evidence — see Branch and Evans (2006), Adam (2007), Hommes (2014), and Pfajfar and Žakelj (2014), who show that very simple AR(1) rules might be used by agents to forecast inflation in survey and experimental settings. Several papers that estimate DSGE models with adaptive expectations, for example, Slobodyan and Wouters (2012) and Ormeno and Molnar (2015), show that assuming agents use very simple forecasting rules leads to superior model fit.

We model our agents as econometricians who do not have prior knowledge about the underlying structure of the economy. However, they do the best they can using the past data. In order for this learning process to converge, three conditions must hold. First, for the agents' PLM (5) to be the equilibrium solution, coefficient b_1 must be equal to the OLS regression coefficient:

$$b_1 = \frac{Cov(y_t, w_t^1)}{Var\left(w_t^1\right)}. \tag{10}$$

Second, the equilibrium (5) must be expectationally stable (E-stable). Finally, the forecast errors produced by their rule of choice, (5), must be smaller than those of the alternative, (6). We compare forecast errors using mean squared forecast errors (MSFE).

Proposition 2.0.1. *In RPE1 (RPE2), where the agents use equation 5 (6) as the forecasting rule, the equilibrium coefficient* $b_1 (b_2)$ *is given by*

$$\begin{aligned} b_1 &= \frac{\gamma_1 + \gamma_2 \rho}{1 - \beta \rho_1}, \\ b_2 &= \left(\beta \rho_1 b_1 + \gamma_1\right) \rho + \gamma_2. \end{aligned}$$

Both RPE1 and RPE2 are E-stable. The MSFE for an agent inhabiting RPE1 and using (5) as the forecasting rule, $MSFE_1$*, is smaller than the MSFE of an agent using (6),* $MSFE_2$*, and thus RPE1 is an equilibrium if:*

$$b_1^2 > b_2^2. \tag{11}$$

Proof. Appendix A. □

Note that in the case of non-standardized variables, condition (11) will be re-written as $\widetilde{b}_1^2 \sigma_1^2 > \widetilde{b}_2^2 \sigma_2^2$, where $\widetilde{b}_{1,2}$ are related to $b_{1,2}$ in the standardized case as $\widetilde{b}_i = b_i \frac{\sigma_y}{\sigma_i}$.

We next allow the agents to challenge their equilibrium forecasting rules and possibly reconsider them. We conduct our analysis for the case where RPE1 is an equilibrium. That is, the agents have initially chosen to use w^1 in their forecasting rule and have found out ex post that it produces smaller forecast errors than the rule with w^2, i.e., (11) holds. Note that in our simple model, both forecasting rules are under-specified. Thus, the analysis for the case where the agents have initially started with RPE2 will be symmetric.

Our agents know that there are other variables in RPE1 (7), and w_t^2 is observable. They also know that using w_t^2 alone for forecasting is inferior to using only w_t^1, because (11) is true. However, they may wonder whether *adding* w_t^2 to their forecasting rule is beneficial. The forecasting rule that includes both w_t^1 and w_t^2 would be clearly superior in this model if the agents were allowed to learn its coefficients, coinciding with $\bar{b}_1$ and $\bar{b}_2$.[2] The agents, however, are subject to attention cost, modeled as in Gabaix (2014). They could attach weights to a variable according to its importance. The importance of a variable depends on its contribution to the variance of the variable of interest (y in our case) and to the agents' utility. The weights then determine how much attention is paid to

[2] If *all* the agents start using both shocks in their forecasting rule, using least-squares learning from then onward, then their PLM coefficients will converge to those of MSV solution (3), (4), as was shown in Evans and Honkapohja (1994). However, we are asking whether an *atomistic* agent could find it advantageous to use both shocks for forecasting.

a variable given the exogenous cost of attention and the loss stemming from inattention, which is a reduction in the quality of the forecast. We let the agents choose the attention vector by maximizing the precision of their forecast of y_t as in (7):

$$u = -\frac{1}{2}\left(\hat{y}_t - y_t\right)^2. \tag{12}$$

For rational agents capable of paying attention to both w^1 and w^2, the optimal forecast is equal to $\hat{y}_t = \widehat{\bar{b}}_1 w_t^1 + \widehat{\bar{b}}_2 w_t^2$, where $\widehat{\bar{b}}_1$ and $\widehat{\bar{b}}_2$ are OLS estimates of the coefficients in (7). That is, fully informed agents with RE use both shocks in the forecasting rule. Sparse rational agents face a trade-off between the attention cost and the increase in forecast precision. They optimize by allocating attention between the variables and forming their optimal forecast rule as $\hat{y}_t = m_1 \cdot \widehat{\bar{b}}_1 w_t^1 + m_2 \cdot \widehat{\bar{b}}_2 w_t^2$, where m_1, $m_2 \in [0,1]$ are attention weights, and $\widehat{\bar{b}}_1$, $\widehat{\bar{b}}_2$ are OLS estimates of the coefficients in the selected forecasting rule. As in Gabaix (2014), we let the agents use the Lasso estimator to derive the optimal attention vector $m = (m_1, m_2)^{'}$, where the attention cost becomes the Lasso penalty term:

$$m = \arg \min_{m \in [0,1]^2} \frac{1}{2} \sum_{i,j=1\ldots2} (1-m_i)\Lambda_{ij}\left(1-m_j\right) + \kappa \sum_{i,j=1\ldots2} |m_i|. \tag{13}$$

The loss arising from inattention, given inattention vector $1-m$, is given by the quadratic form $(1-m)^{'}\Lambda(1-m)$, with $[\Lambda]_{ij} = -\sigma_{ij} a_{w_i} u_{aa} a_{w_j}$. This loss reflects how much of the variation in the process we lose when (partially) neglecting variables. σ_{ij} is equal to Σ_{ij}^w, $a_{w_i} = -u_{aa}^{-1} u_{aw_i}$ determines by how much a change in a variable w_i changes the agent's action a, equal to the forecast $\hat{y}_t$. Finally, the parameter κ governs the attention cost. The derivatives of the agents' utility (12) are given by

$$\begin{aligned} u_{aa} &= \frac{\partial^2 u_a}{\partial a^2} = \frac{\partial}{\partial a}\left(-(a - \bar{b}_1 w_t^1 - \bar{b}_2 w_t^2)\right) = -1, \\ u_{aw_i} &= \bar{b}_i. \end{aligned}$$

Then $a_{w_i} = \bar{b}_i$, and the cost of inattention is therefore given by a quadratic form Λ, with $[\Lambda]_{ij} = \sigma_{ij}\bar{b}_i\bar{b}_j$.

In the interior solution, taking the derivatives of (13) with respect to m_1 and m_2 gives the following expressions (details and corner solutions are presented in Appendix A.3):

$$m_1 = 1 - \frac{\kappa}{\bar{b}_1\bar{b}_2\left(1-\rho^2\right)} \frac{\bar{b}_2 - \rho\bar{b}_1}{\bar{b}_1}, \tag{14}$$

$$m_2 = 1 - \frac{\kappa}{\bar{b}_1\bar{b}_2\left(1-\rho^2\right)} \frac{\bar{b}_1 - \rho\bar{b}_2}{\bar{b}_2}. \tag{15}$$

One can immediately observe that both weights are falling with attention cost, κ. If the attention cost is large enough, agents choose not to pay attention to any variable, and use only a constant term in their forecasts. Both weights are decreasing in the ALM coefficient on another variable: e.g., the weight on the first shock is lower if $\bar{b}_2$ is larger.

Proposition 2.0.2. *The condition for $m_1 > m_2$ coincides with the condition for $MSFE_1 < MSFE_2$:*

$$\bar{b}_1^2 > \bar{b}_2^2,$$

which is equivalent to condition (11).

Proof. Appendix A.4 □

Proposition 2.0.2 states that as long as using the RPE1-consistent PLM (5) produces smaller forecast errors than using PLM (6), sparsely rational agents optimally pay more attention to w_t^1 than to w_t^2.

If the weight on shock w_t^2 is positive, and all agents continue learning, as described in the footnote (2), eventually the system will converge to the RE MSV solution. Therefore, in order for the agents to stick to the mis-specified rule (5), m_2, the weight on shock w_t^2, must be zero.

Proposition 2.0.3. *The second shock gets a non-zero weight in the agents' forecast when:*

$$\kappa \leq \frac{\left(1-\rho^2\right)\bar{b}_2^2}{1-\rho\frac{\bar{b}_2}{\bar{b}_1}}. \tag{16}$$

Proof. Re-arranging (15), we get

$$\frac{\kappa}{\bar{b}_1\bar{b}_2\left(1-\rho^2\right)}\frac{\bar{b}_1-\rho\bar{b}_2}{\bar{b}_2} \leq 1 \Rightarrow \kappa \leq \frac{\left(1-\rho^2\right)\bar{b}_2^2}{1-\rho\frac{\bar{b}_2}{\bar{b}_1}}.$$

Appendix A.3 shows that if a solution with the nonzero weight m_2 exists, it outperforms all corner solutions with $m_2 = 0$. Therefore, (16) is sufficient for the solution with nonzero m_2 (and thus m_1) to exist. □

Proposition 2.0.3 also states that if the RPE1-consistent rule is an equilibrium, that is, if (11) is satisfied, there is no region in the parameter space such that RPE1 survives for any κ, as for sufficiently small κ (16) guarantees $m_2 \geq 0$.

We can interpret condition (16) as follows. Consider the ALM (7). It includes two normally distributed variables, $\tilde{w}_t^1 = \bar{b}_1 w_t^1$ and $\tilde{w}_t^2 = \bar{b}_2 w_t^2$. The correlation coefficient between $\tilde{w}_t^1$ and $\tilde{w}_t^2$ is equal to ρ. Then $\rho\frac{\bar{b}_2}{\bar{b}_1}$ represents the coefficient in a regression of $\tilde{w}_t^2$ on $\tilde{w}_t^1$, while $\left(1-\rho^2\right)\bar{b}_2^2$ is the variance of the conditional distribution of $\tilde{w}_t^2$ given $\tilde{w}_t^1$. Thus, if the cost of attention, κ, corrected for the information about $\tilde{w}_t^2$ already contained in (5), is larger than the variance of the omitted information — the variance of $\tilde{w}_t^2$ conditional on $\tilde{w}_t^1$, then the agents find it beneficial not to include the second shock in their forecasts. Condition (16), then, has an intuitive economic interpretation as equalizing the cost and benefits of considering that portion of the information contained in the second shock w_t^2 which is above and beyond that which is already evaluated, given that w_t^1 is taken into account in the forecast.

For a more intuitive representation, we now introduce a measure which is affected not by the absolute values of the shock variances and persistences, but only by their ratio. For this purpose, we normalize the attention cost in (16) by $\bar{b}_2^2 = \gamma_2$. We define the cost-to-variance ratio as $f \equiv \frac{\bar{\kappa}}{\bar{b}_2^2} = \frac{1-\rho^2}{1-\rho\bar{b}_2/\bar{b}_1}$, where $\bar{\kappa}$ is the threshold for the second shock to be included in the agents' rule. For $\kappa < \bar{\kappa}$, the agents use both shocks in their forecasting rules, and for $\kappa \geq \bar{\kappa}$ the agents stick to the rule with w^1 only. We plot this ratio f for different values of individual shocks persistencies, ρ_i, expressing it in terms of original, non-standardized, parameters independent of ρ_i's, that is, as a function of non-standardized γ and of standard deviations to *i.i.d.* innovations to the shocks: σ_{ε_i}. Figure 1 presents the plot of f in the coordinates $(\rho, \log(\Gamma\Sigma))$, where $\Gamma\Sigma \equiv \frac{\gamma_1\sigma_{\varepsilon_1}}{\gamma_2\sigma_{e_2}}$ is the relative importance of the innovation to the first shock, ε_t^1, in the true data-generating process (1). We do this for different values of ρ_1 and ρ_2. The agents include w_t^2 in the forecasting rule when the normalized attention cost is smaller than f. Thus, larger f (lighter shade) means that the range of costs consistent with w_t^2 used for forecasting is wider.

In Figure 1, the white area corresponds to the parameter values where PLM (5) is not selected, as it produces forecast error larger than PLM (6). The hatched area defines the region where a particular combination of persistence of shocks and the correlation between them is impossible.[3] A larger persistence of w_t^2, ρ_2, reduces the parameter space where $MSFE_1 < MSFE_2$, while increasing ρ_1 expands this area.

A large absolute value of correlation between the shocks, $|\rho|$, also contributes to better forecasting performance of the RPE1-consistent PLM (A1), so that $MSFE_1 < MSFE_2$. As described in Appendix A.2, the $MSFE_1 < MSFE_2$ condition is satisfied for $|\rho| > \frac{1-\beta\rho_1-\Gamma}{\beta\rho_1}$. When the MSFE criterion (11) is satisfied, a large absolute correlation increases the variation in y_t explained by the first shock alone, making the right-hand side of (16) smaller.

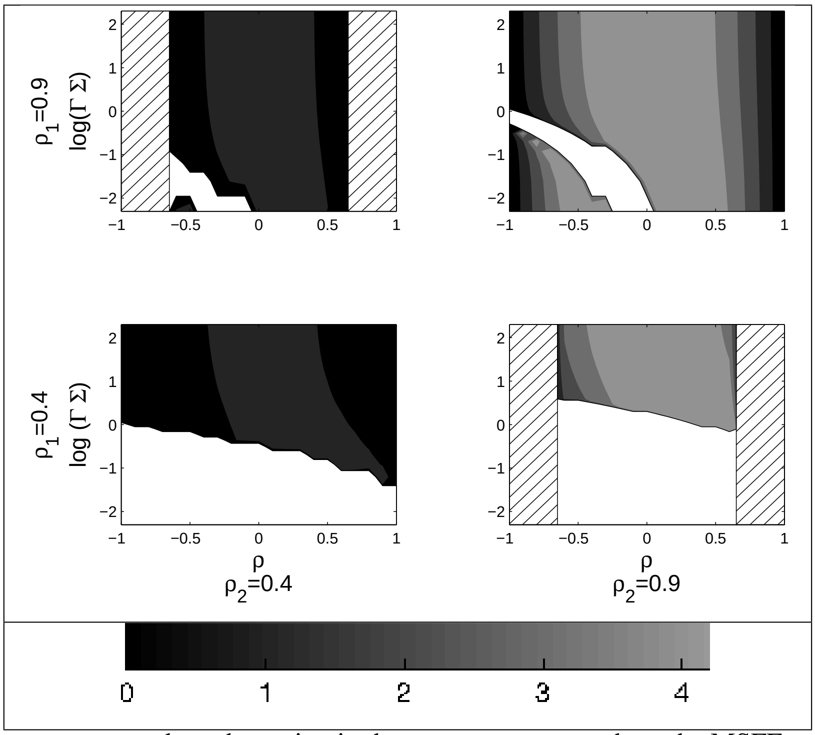
Notice that a positive correlation, ρ, leads to a positive m_2 for larger $\frac{\kappa}{\bar{b}_2^2\sigma_{\varepsilon 2}^2}$ than a negative correlation with the same absolute value. The asymmetric effect of the correlation between w^1 and w^2 is explained in Figure 2, where we plot the inattention cost, the first term on the RHS in equation (13), as an ellipse, and the corresponding attention cost, the second term on the RHS in equation (13), as a line, for different values of correlation, ρ. As shown in Appendix A.5, the slope of the major axis of the ellipse depends on the sign of ρ directly and through the term $\bar{b}_1$. As ρ increases, the ellipse rotates and generally expands, but the attention cost is not affected; thus the ellipse and the isocost line touch in different places depending on the sign of the correlation. As is obvious from the figure, for positive ρ optimality is achieved at higher (m_1, m_2), because the inattention cost is generally higher in the north-east quadrant of Figure 2 for higher ρ.

Finally, note that even when $\log(\Gamma\Sigma) > 0$ (and $\Gamma\Sigma > 1$), so that w_t^1 plays a more important role in explaining the variation in (1) than w_t^2, and both ρ_1 and ρ are large, so that taking into account w_t^1 is very informative (see the upper right panel of Figure 1), there could still be an attention cost low enough for w_t^2 to be included in sparsely rational agents' PLM. Therefore, informationally unconstrained agents with $\kappa = 0$ will always find it beneficial to include w_t^2 in their forecasting rules.

In the next section we move to a three-equation New Keynesian model and study possible policy implications.

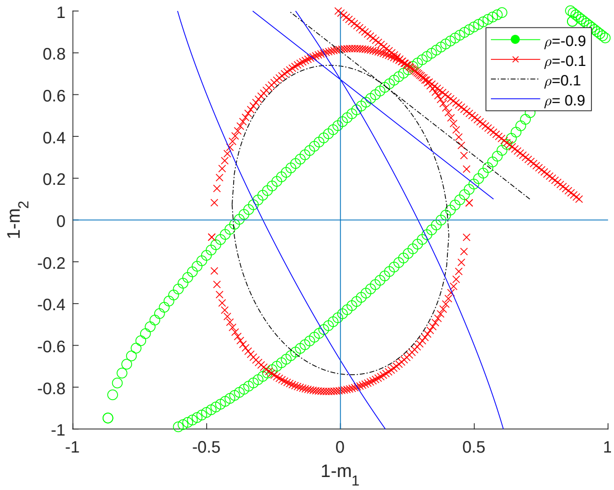
[3] The correlation between w_t^2 and w_t^1 can be expressed as $\rho = \frac{Cov(w^1,w^2)}{\sigma_1\sigma_2} = \rho_{\varepsilon_{12}}\frac{\sqrt{(1-\rho_1^2)(1-\rho_2^2)}}{1-\rho_1\rho_2}$, where $|\rho_{\varepsilon_{12}}| < 1$ is the correlation between the innovations to the shocks. This implies that $|\rho| < |\frac{\sqrt{(1-\rho_1^2)(1-\rho_2^2)}}{1-\rho_1^2\rho_2}|$.

Figure 1: Threshold for the Cost-to-Variance Ratio



Note: The white area corresponds to the region in the parameter space where the MSFE condition is not satisfied for RPE1. The hatched area corresponds to the region where a particular combination of shock persistence and correlation is impossible due to $|\rho| \leq 1$.

Figure 2: Inattention Cost as a Function of Correlation, ρ.



Note: The numerical values used for the plot are $\kappa = 0.1$, $\rho_1 = \rho_2 = 0.4$.

3. Three-Equation New Keynesian Model

Having studied the choice of forecasting rules when all the variables are exogenous, in this section we move to a New Keynesian model with endogenous variables. We employ the textbook three-equation New Keynesian model (cf., Galí 2015, section 3), but modify it to include external habit formation. We first show that the conditions derived in the previous section can be generalized to this model, and then discuss the policy implications.

As shown in Appendix B, the equations of the model linearized around the deterministic steady state, where we have assumed an extreme case of habit persistence, take the form:

$$y_t = y_{t-1} - \frac{1}{\sigma}(i_t - E_t\pi_{t+1}) + g_t, \quad (17)$$

$$\pi_t = \beta E_t\pi_{t+1} + \omega y_t + u_t, \quad (18)$$

where π_t is inflation and y_t the output gap, g_t and u_t are Gaussian shocks with finite variance, and σ and ω, and β are standard coefficients calibrated as in Galí 2015, section 3. We assume that the households do not observe the realization of current shocks and treat them as zero mean random variables: $E_t(g_t) = E_t(u_t) = 0$.

The monetary authority sets interest rates in reaction to households' inflation expectations, which may deviate from rational expectations. We consider two alternative forms of the monetary policy rules, with the monetary authority reacting to the deviation of expectations of the future or the current inflation from the steady state:

$$i_t = \phi_\pi E_t\pi_{t+1}, \quad (19)$$

$$i_t = \phi_\pi E_t\pi_t. \quad (20)$$

The rational expectations equilibria and their properties under different policy rules are formalized in proposition 3.0.1.

Proposition 3.0.1. *Under the forward-looking policy rule, (19), the model becomes:*

$$y_t = \frac{1-\phi_\pi}{\sigma}E_t\pi_{t+1} + y_{t-1} + g_t, \quad (21)$$

$$\pi_t = (\beta + \omega\frac{1-\phi_\pi}{\sigma})E_t\pi_{t+1} + \omega y_{t-1} + \omega g_t + u_t. \quad (22)$$

REE solutions to (21) and (22) of the form $\hat{\pi}_t = b\pi_{t-1} + cy_{t-1} + dg_t + zu_t$*, with the coefficients defined in (B23)–(B26), are not determinate. The minimum state variable solution (MSV) of the form*

$$\hat{\pi}_t = cy_{t-1} + dg_t + zu_t, \quad (23)$$

with the coefficients defined in (B39)–(B42), is both E-stable and determinate, as long as $\phi_\pi > 1$.

Under the rule with contemporaneous inflation expectations, (20), the solution to the model is:

$$y_t = -\frac{\phi_\pi}{\sigma}E_t\pi_t + \frac{1}{\sigma}E_t\pi_{t+1} + y_{t-1} + g_t, \quad (24)$$

$$\pi_t = (\beta + \frac{\omega}{\sigma})E_t\pi_{t+1} - \frac{\omega\phi_\pi}{\sigma}E_t\pi_t + \omega y_{t-1} + \omega g_t + u_t. \quad (25)$$

REE solutions to (25) and (24) of the form $\hat{\pi}_t = b\pi_{t-1} + cy_{t-1} + dg_t + zu_t$, with the coefficients defined in (B69)–(B72), are not determinate. The minimum state variable solution (MSV) of the form $\hat{\pi}_t = cy_{t-1} + dg_t + zu_t$, with the coefficients defined in (B79)–(B81) is both E-stable and determinate, as long as $\phi_\pi > 1$ and $\phi_\pi < 2\frac{\sigma}{c} + 1$.

Proof. The proof is given in Appendix B.2. □

As in the simple model of section 2, we also standardize the variables. As the steady state values of π and y are zero, we simply divide them by their standard deviations. The coefficients in the Phillips curve (18) are multiplied by $\frac{\sigma_x}{\sigma_\pi}$, where σ_x is the standard deviation of variable x. In the IS equation (17), all the coefficients are multiplied by $\frac{\sigma_x}{\sigma_y}$. The covariance between the standardized variables is obtained as $Cov(y,\pi) = \frac{Cov(y,\pi)}{\sigma_y \sigma_\pi}$. To economize on notation, we continue using symbols π and y while referring to these variables in their standardized form, unless stated otherwise.

As in the previous section and in Adam (2005), we restrict our agents to using only one (endogenous) variable in their forecasting rules (as the shocks u_t and g_t are unobservable, they cannot be used). Accordingly, we consider two RPEs in which agents use either lagged inflation or the lagged output gap in their forecasting rules. In the first RPE, the agents use as their PLM

$$\hat{\pi}_t = \alpha_\pi + \beta_\pi \widetilde{\pi}_{t-1}, \tag{26}$$

while in the other one the rule is

$$\hat{\pi}_t = \alpha_\pi^y + c_\pi^y \widetilde{y}_{t-1}. \tag{27}$$

We call these PLMs M_π and M_y, respectively. We also call the resulting RPEs M_π and M_y. In the context of our model, the rule M_y includes the 'correct' endogenous variable y, which belongs to the MSV solution (23), while M_π does not, as π is not present in (23). However, if the agents stick to M_π, the lag of inflation affects the actual law of motion through the expectational terms, and the actual law of motion is given by (B87). For this rule to be an equilibrium choice, it should result in smaller forecast errors than the alternative one, M_y.

Proposition 3.0.2. *For the model described by (17) and (18), and the policy rule as in (19) or (20), the condition for M_π to be an equilibrium, $MSFE_\pi < MSFE_y$, is:*

$$\bar{b}_\pi^2 > \bar{c}_\pi^2, \tag{28}$$

where $\bar{b}_\pi$ and $\bar{c}_\pi$ are the M_π ALM coefficients on the standardized lag of inflation and the output gap, respectively, given by equations (B87) and (B88).

Proof. The proof is given in Appendix B.3. □

Rewriting (28), we can see that M_π results in smaller mean forecast squared errors than M_y if

$$\bar{\Gamma}^2 > 1, \tag{29}$$

where $\bar{\Gamma} = \frac{\bar{b}_\pi}{\bar{c}_\pi}$. Thus, if in the ALM consistent with M_π RPE, (B87), the share of the inflation variation explained by inflation term is larger than the share accounted for by the output, then M_π results in smaller forecast errors than M_y .

The criterion for M_π to be an equilibrium, (28), is similar to the condition (11) from the simple model: the variable used by the agents in their PLM should be more important than the one that is omitted, for the RPE consistent with this PLM to be an equilibrium.

3.1 Attention Weights

Suppose that the actual law of motion, induced by the M_π rule, is given in (B87)–(B88). The agents now can reconsider their forecasting rules, subject to the attention cost constraint. Will they give a significant weight to the past output? In other words, will they include the correct 'variable' y or stick to the rule with the 'wrong' one, π?[4] Will the agents choose not paying attention to any of the variables? To answer these questions, we let the agents select the sparse weights, a vector $m = \left(m_\pi, m_y\right)'$, by minimizing $(\hat{\pi}_t - \pi_t)^2$ subject to the attention cost. The difference from the simple case in Section 2 is that the variables used for forecasting are now endogenous. We rewrite the utility function as:

$$u = -\frac{1}{2}(m_y \bar{c}_\pi y_{t-1} + m_\pi \bar{b}_\pi \pi_{t-1} - b_\pi \pi_{t-1} - c_\pi y_{t-1})^2. \tag{30}$$

To find the sparse weights, the agents minimize (13), where the quadratic form for the cost of inattention is given by $\Lambda_{ij} = -\sigma_{ij} a_{w_i} u_{aa} a_{w_j}$. The corresponding derivatives are given below:

$$\begin{aligned}
u_{aa} &= \frac{\partial^2 u_a}{\partial a^2} = \frac{\partial}{\partial a}\left(-(a - \bar{b}_\pi \pi_{t-1} - \bar{c}_\pi y_{t-1})\right) = -1, \\
u_{a\pi_{t-1}} &= \bar{b}_\pi, \\
u_{ay_{t-1}} &= \bar{c}_\pi, \\
a_{\pi_{t-1}} &= -u_{aa} u_{a\pi_{t-1}} = -\bar{b}_\pi, \\
a_{y_{t-1}} &= -u_{aa} u_{ay_{t-1}} = -\bar{c}_\pi.
\end{aligned}$$

Then the cost of inattention is given by $(1-m)' \Lambda (1-m)$ with $\Lambda = \begin{pmatrix} \bar{b}_\pi^2 & \sigma_{\pi y} \bar{b}_\pi \bar{c}_\pi \\ \sigma_{\pi y} \bar{b}_\pi \bar{c}_\pi & \bar{c}_\pi^2 \end{pmatrix}$, where $\sigma_{\pi y}$ is the covariance of the standardized output gap and inflation, derived in Appendix B.3.

Taking the first-order conditions of (13) and solving for weights results in the following expressions:

$$m_y = 1 - \frac{\kappa}{\bar{c}_\pi^2\left(1-R^2\right)} \frac{\bar{b}_\pi - \bar{c}_\pi R}{\bar{b}_\pi}, \tag{31}$$

$$m_\pi = 1 - \frac{\kappa}{\bar{b}_\pi^2\left(1-R^2\right)} \frac{\bar{c}_\pi - \bar{b}_\pi R}{\bar{c}_\pi}, \tag{32}$$

where R is the correlation between y_t and π_t in RPE_π, the equilibrium consistent with agents using the M_π rule.

Proposition 3.0.3. *The condition for $m_\pi > m_y$ coincides with the condition for $MSFE_\pi < MSFE_y$:*

$$\bar{b}_\pi^2 > \bar{c}_\pi^2.$$

Proof. The proof is given in Appendix C. □

[4] Again, if the agents start to include lagged output in their forecasting rules and employ least-squares learning from then onward, they eventually learn the coefficient on past π and y, with the former being zero.

The next proposition summarizes the condition for output to get a positive weight in agents' forecast.

Proposition 3.0.4. *The lag of output gets a positive weight in agents' forecast when:*

$$\kappa \leq \frac{\left(1-R^2\right)\bar{c}_\pi^2}{1-R\frac{\bar{c}_\pi}{\bar{b}_\pi}}. \tag{33}$$

Proof. The proof is given in Appendix C. □

Notice the similarity of (33) with (16).

Propositions 3.0.2–3.0.4 have established that the conditions guaranteeing the existence of M_π RPE, the relative importance of the weights on inflation and the output gap, and the condition of a non-zero weight on the output gap all look very similar to the simple model case of Section 2. However, in the New Keynesian model monetary policy can affect the joint dynamics of π_t and y_t, which generates additional insights.

3.2 Policy Rules and Forecasting Rules

Next we consider the interaction between monetary policy and the equilibrium selection of forecasting rules. For all numerical simulations we use the textbook calibration with $\beta = 0.99$, the risk aversion coefficient $\sigma = 1$, the Frisch elasticity of labor supply $\phi = 1$, and Calvo probability $\theta = 2/3$.[5] For some of the graphs we use the ratio of the original shock deviations, $r \equiv \frac{\sigma_u}{\sigma_g}$, where u and g are non-standardized innovations in the IS and Phillips curves, respectively, cf. (17) and (18). Without loss of generality, we set $\sigma_g = 1$ in our simulations.

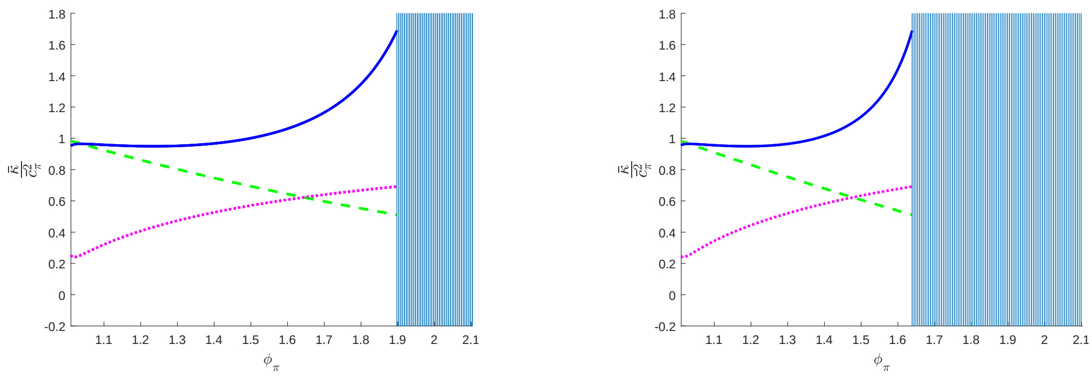
Figure 3 shows the dimensionless threshold for the cost-to-variance ratio for both policy rules, defined as:

$$f \equiv \frac{\bar{\kappa}}{\bar{c}_\pi^2} = \frac{\left(1-R^2\right)}{1-R\frac{\bar{c}_\pi}{\bar{b}_\pi}}. \tag{34}$$

Here, $\bar{\kappa}$ is the maximum cost of attention at which the agents are willing to include the output gap into their forecasting rules. Figure 3 shows that when attention cost is present, the forecasting rule with the 'wrong' variable, M_π, can be supported under sparse rationality. With very aggressive policy (ϕ_π close to 2), the model with the incorrect variable is never the equilibrium choice, as it produces a larger MSFE than M_y. This region is `colored blue` and shaded in Figure 3. In the parameter region where the rule M_π is chosen, the stronger is the policy reaction, the larger is the range of attention cost consistent with the lagged output gap being included in the forecasting rule. The agents choose to include both the output gap and inflation for larger attention cost when the monetary policy is more aggressive. This is evident from the larger area under the threshold for the cost-to-variance, represented by the solid line. The intuition for this result can be found in the ALM coefficients. The coefficient on past inflation in the ALM for inflation, $\bar{b}_\pi$, represented by the dashed line, is decreasing as ϕ_π increases, meaning that inflation becomes predicted better by the lagged output gap and worse by its own past value. With an increasing importance of the output gap in predicting inflation, it is not surprising that the agents agree to pay more to include the output gap in their forecasting rules.

[5] These values are taken from Galí 2015, section 3.

Figure 3: Threshold for the Cost-to-Variance Ratio



(a) Forward-Looking Rule ***(b) Contemporaneous Rule***

Note: The dotted line corresponds to the correlation between output and inflation, the dashed line to $\bar{b}_\pi$, the ALM coefficient on past inflation, and the blue solid line to f, the threshold for the cost-to-variance ratio. The blue-colored region shows the area where $\text{MSFE}_\pi > \text{MSFE}_y$. For the numerical plot we set $r \equiv \frac{\sigma_u}{\sigma_g} = 0.1$, $\sigma_g = 1$.

The correlation between the output gap and inflation (the dotted line in Figure 3) is increasing with policy aggressiveness. As was discussed in Section 2, the effect of the correlation on the weights is not linear. When inflation is persistent (the coefficient on its lag is large in the ALM) and volatile, a larger correlation contributes to a smaller weight on the output gap. When inflation becomes less persistent and volatile, the agents assign a larger weight to the output gap with a larger correlation.

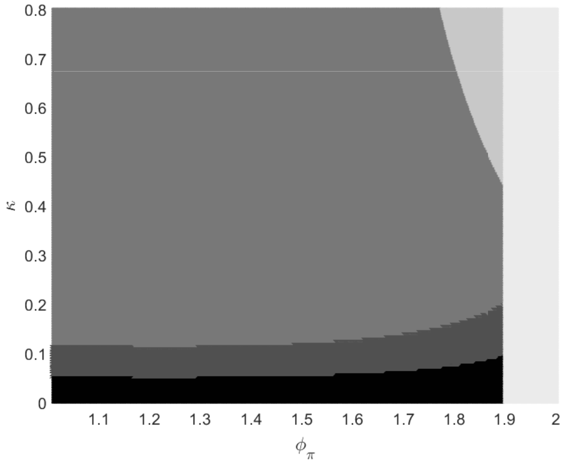
The two panels of Figure 3 are rather similar, demonstrating that there is no significant difference between the contemporaneous data-based and expectations-based policy rules. Under the contemporaneous policy rule, (20), the PLM with the 'wrong' variable, M_π, can be avoided with less aggressive monetary policy, as for any ϕ_π, the threshold attention cost-to-variance ratio is larger; the agents therefore find it more important to pay attention to output with rule (20).

Figure 4 shows the weights m_i the agents choose for different values of the learning cost, κ, and the policy parameter, ϕ_π, while fixing the ratio of the deviations in the innovations to $r = 0.1$.[6] When the weight on the output gap is zero but the weight on inflation is positive, the region is denoted as M_π, as it corresponds to the agents sticking to the rule with the inflation lag. For large attention cost, κ, and large ϕ_π, the agents choose to have zero weights on both variables and use only the constant in their forecasting rules. This region in the upper-right corner of Figure 4 is denoted as (0,0). This only happens for very aggressive monetary policy, where the volatility of both inflation and output is very small, as shown in Figure 5.

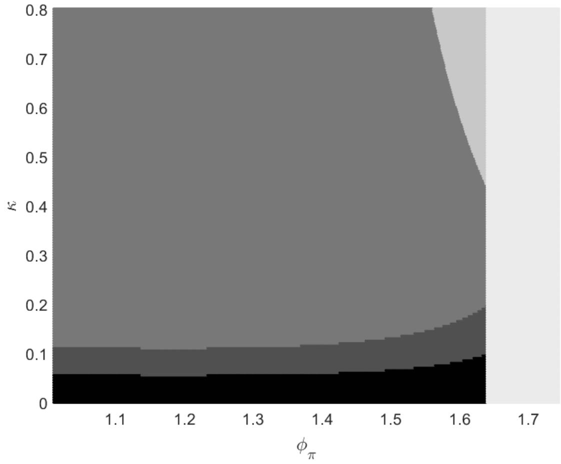
As monetary policy becomes more aggressive, ϕ_π increases, the weight on output starts to rise, and the area denoted by M_π gets smaller. Thus, when the policy response to inflation is stronger, the agents switch to a 'correct' rule for a larger parameter set. Figure D1 in Appendix D presents the graph, similar to Figure 4, which plots m_π as a function of (ϕ_π, κ).

[6] With the standard deviation of the output shock σ_g fixed, larger r means larger relative inflation volatility. This results in smaller estimates of the coefficient on inflation and smaller weights. With larger r, Figure 4 and Figure D1 look similar, but the agents move to REE ($m_y > 0$) with a smaller policy parameter ϕ_π.

Figure 4: Model Selection under Sparse Weights, M_y



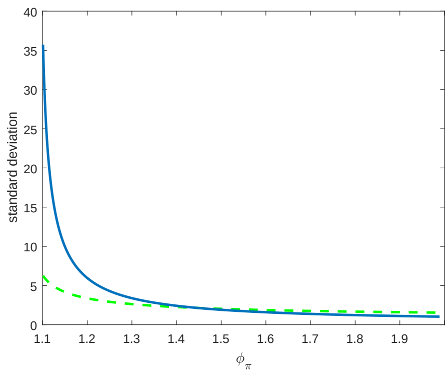
(a) Forward-Looking Rule



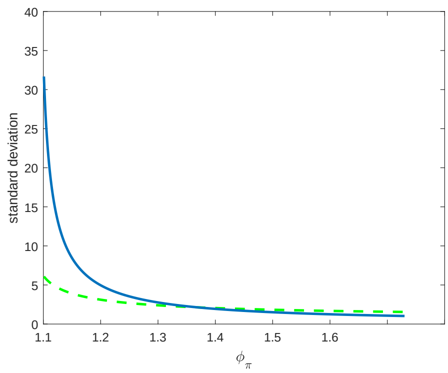
(b) Contemporaneous Rule

Note: For the numerical plot we set $r \equiv \frac{\sigma_u}{\sigma_g} = 0.1$, $\sigma_g = 1$.

Figure 5: Equilibrium Dynamics of Inflation and Output



(a) Forward-Looking Rule



(b) Contemporaneous Rule

Note: The dashed line corresponds to the output gap and the solid line to inflation. For the numerical plot we set $r \equiv \frac{\sigma_u}{\sigma_g} = 0.1$, $\sigma_g = 1$.

For any strength of the policy response, the rule with contemporaneous inflation results in less overall volatility. This finding is consistent with the experimental results in Pfajfar and Žakelj (2018), where the contemporaneous rule is found to produce smaller volatility than the forward-looking rule. The large inflation volatility in Figure 5 can be linked to large output gap forecast errors.

Our results contribute to the literature discussing how monetary policy affects not only the level of expectations, but also the way expectations are formed. In our model, more restrictive monetary policy reduces the parameter range of stability of the M_π rule in favor of both the M_y rule, with the correct variable, and the intercept-only rule: for large ϕ_π the agents either start taking into account the output gap if κ is small or disregard all variables if κ is larger. The result is intuitive, since in the model we study, the strength of the policy, ϕ_π, reduces the feedback from expectations in the actual law of motion.[7] The role of the feedback parameter is widely studied in the literature. In an experimental setting, Hommes (2014) and Heemeijer et al. (2009) emphasize the importance of the expectations feedback parameter. In Hommes (2014), when the expectations feedback parameter is negative, convergence to REE is observed, but with a positive parameter agents coordinate on a non-rational self-fulfilling equilibrium. A large feedback parameter results in convergence to a mis-specified equilibrium in Evans et al. (2012). In the laboratory experiments of Pfajfar and Žakelj (2014), Pfajfar and Žakelj (2018), and Assenza et al. (2013), agents choose forecasting rules for inflation under alternative monetary policy regimes differing in terms of the aggressiveness of the response to inflation in the Taylor rule. As monetary policy becomes more aggressive, more agents switch to using forecasting rules compatible with rational expectations. This supports our finding that for low κ and large ϕ_π agents increase their weight on the lagged output gap.

Another result of this paper is that agents stick to the AR(1) model for inflation forecasting when the persistence of inflation is high and/or the correlation between output and inflation is low (see Figure 3, low values of ϕ_π). This result is in line with the observed behavior of professional forecasters after the recent financial crisis. There are a number of studies, examples being Fendel et al. (2011), Lopez-Perez (2017), and Frenkel et al. (2011) showing that professional forecasters' predictions behave as if they were using the Phillips curve. After the financial crisis, inflation became more persistent – see Watson (2014) – while the Phillips curve got flatter.[8] In terms of our model, this means a lower threshold for the cost-to-variance ratio in Figure 3. Lopez-Perez (2017)[9] shows that forecasters' predictions started to react much less to unemployment after the financial crisis of 2007–2009, consistent with our model predictions.

4. Conclusion

In this paper we study whether an initially mis-specified forecasting rule which generates a restricted perceptions equilibrium (RPE) can be an equilibrium choice under sparse rationality, thus perpetuating the RPE if attention is costly. We first consider a simple process consisting only of exogenous variables, and then generalize our results to a three-equation New Keynesian model with lagged endogenous variables arising due to agents' expectations.

[7] To see this, consider the coefficient on inflation expectations in the actual law of motion for inflation in (B9) and (B56) for the forward-looking and contemporaneous rules, respectively.

[8] Although the evidence on the flattening of the Phillips curve is mixed due to the different specifications and time horizons considered, there are studies showing a decline in slope, examples being IMF (2013) and Kuttner and Robinson (2010). Donayre and Panovska (2016) document breaks in the wage Phillips curve during recessions and subsequent recoveries.

[9] Frenkel et al. (2011) uses data up to 2010Q and does not find evidence for a change in forecasters' behavior, while Lopez-Perez (2017) uses a longer data set and includes a forward-looking inflation term in the Phillips curve.

For both models we find regions in the parameter space, where a RPE with the variable not present in the MSV RE solution is selected by both the minimum squared forecast error condition and sparse-rationality considerations. Thus, agents who are reconsidering their initial choice of a mis-specified RPE could choose to continue using the initial rule. If attention cost is very large, there is a region of the parameter space where agents choose not to allocate attention to any of the variables. If attention cost is small, agents tend to switch to the RPE with the variable that is present in the MSV RE solution, and do this certainly when the attention cost is zero. For a medium range of attention costs, the initial mis-specified forecasting rule prevails for large persistence of the variable used in the rule and for large correlation between the included and the omitted variable, especially if the omitted variable has low persistence. This behavior is explained by the amount of additional information that is contained in the omitted variable.

In the New Keynesian model we find that when inflation persistence increases, survival of the mis-specified AR(1) rule for inflation forecasting is achieved for a larger region of the parameter space. The same is true for smaller correlation between inflation and the output gap. This prediction is reminiscent of the behavior of professional forecasters, whose predictions are found to be consistent with paying less attention to the output gap after the financial crisis (Lopez-Perez 2017), when inflation became more persistent and the correlation between output and inflation might have changed.

We further find that the aggressiveness of the monetary policy rule and, to some extent, the rule itself, determine the survival of a mis-specified forecasting rule. A strong monetary policy reaction reduces inflation volatility and persistence, making inflation in the forecasting rule less useful. With an even stronger monetary policy reaction, agents do not consider the mis-specified forecasting rule in the first place, because the rule with the output gap, corresponding to the MSV solution, results in smaller forecast errors. In line with the previous literature, our study supports the importance of the expectations feedback parameter for the survival of a mis-specified rule. In our model, the feedback parameter is decreasing with the policy rule parameter. If the expectations feedback is large enough, the mis-specified forecasting rule may prevail in equilibrium.

As was shown in Slobodyan et al. (2016), monetary policy satisfying E-stability principles under least-squares learning can lead to instability under alternative learning specifications, Pfajfar and Žakelj (2018) shows that with stronger monetary policy, more agents switch to rules consistent with the rational expectations hypothesis. Adding to this literature, our study suggests that providing conditions for the survival of a particular equilibrium when the agents face attention costs can be viewed as an additional criterion for the choice of monetary policy.

References

ADAM, K. (2005): "Learning to forecast and cyclical behavior of output and inflation." *Macroeconomic Dynamics,* 9(01):1–27.

ADAM, K. (2007): "Experimental evidence on the persistence of output and inflation." *Economic Journal,* 117(520):603–636.

ASSENZA, T., P. HEEMEIJER, C. HOMMES, AND D. MASSARO (2013): "Individual expectations and aggregate macro behavior." Tinbergen Institute Discussion Papers 13-016/II, Tinbergen Institute

BRANCH, W. A. AND G. W. EVANS (2006): "Intrinsic heterogeneity in expectation formation." *Journal of Economic Theory,* 127(1):264–295.

DONAYRE, L. AND I. PANOVSKA (2016): "Nonlinearities in the U.S. wage Phillips curve." *Journal of Macroeconomics,* 48(C):19–43.

EVANS, G. W. AND S. HONKAPOHJA (1994): "Learning, convergence, and stability with multiple rational expectations equilibria." *European Economic Review,* 38(5):1071–1098.

EVANS, G. W. AND S. HONKAPOHJA (2009): "Learning and macroeconomics." *Annual Review of Economics,* 1(1):421–451.

EVANS, G. W. AND S. HONKAPOHJA (2001): *Learning and expectations in macroeconomics.* Princeton University Press, Princeton.

EVANS, G. W., S. HONKAPOHJA, T. SARGENT, AND N. WILLIAMS (2012): "Bayesian model averaging, learning and model selection." CDMA Working Paper Series 201203, Centre for Dynamic Macroeconomic Analysis

FENDEL, R., E. M. LIS, AND J. C. RÜLKE (2011): "Do professional forecasters believe in the Phillips curve? Evidence from the G7 countries." *Journal of Forecasting,* 30(2):268–287.

FRENKEL, M., E. M. LIS, AND J. C. RÜLKE (2011): "Has the economic crisis of 2007-2009 changed the expectation formation process in the Euro area?." *Economic Modelling,* 28(4): 1808–1814.

GABAIX, X. (2014): "A sparsity-based model of bounded rationality." *The Quarterly Journal of Economics,* 129(4):1661–1710.

GABAIX, X. (2016): "A behavioral New Keynesian model." Working Paper 22954, National Bureau of Economic Research

GALÍ, J. (2015): *Monetary policy, inflation, and the business cycle: an introduction to the New Keynesian framework and its applications second edition.* Princeton University Press, 2 edition.

GARCÍA-SCHMIDT, M. AND M. WOODFORD (2015): "Are low interest rates deflationary? a paradox of perfect-foresight analysis." NBER Working Papers 21614, National Bureau of Economic Research, Inc

GEFANG, D. (2014): "Bayesian doubly adaptive elastic-net Lasso for VAR shrinkage." *International Journal of Forecasting,* 30(1):1–11.

HEEMEIJER, P., C. HOMMES, J. SONNEMANS, AND J. TUINSTRA (2009): "Price stability and volatility in markets with positive and negative expectations feedback: an experimental investigation." *Journal of Economic Dynamics and Control,* 33(5):1052–1072.

HOMMES, C., D. MASSARO, AND M. WEBER (2017): "Monetary policy under behavioral expectations: theory and experiment." Bank of Lithuania Working Paper Series 42, Bank of Lithuania

HOMMES, C. H. (2014): "Behaviorally rational expectations and almost self-fulfilling equilibria." *Review of Behavioral Economics,* 1(1-2):75–97.

IMF (2013): *The dog that didnt bark: has inflation been muzzled or was it just sleeping?* In *World Economic Outlook,* chapter 3, pages 1–17..

KUTTNER, K. AND T. ROBINSON (2010): "Understanding the flattening Phillips curve." *North American Journal of Economics and Finance,* 21(2):110–125.

LOPEZ-PEREZ, V. (2017): "Do professional forecasters behave as if they believed in the New Keynesian Phillips Curve for the euro area?" *Empirica,* 44(1):147–174.

MACKOWIAK, B. AND M. WIEDERHOLT (2009): "Optimal sticky prices under rational inattention." *American Economic Review,* 99(3):769–803.

MANKIW, N. G. AND R. REIS (2007): "Sticky information in general equilibrium." *Journal of the European Economic Association,* 5(2-3):603–613.

MARCET, A. AND T. J. SARGENT (1989): "Convergence of least squares learning mechanisms in self-referential linear stochastic models." *Journal of Economic Theory,* 48(2):337–368.

MATEJKA, F. AND A. MCKAY (2015): "rational inattention to discrete choices: a new foundation for the multinomial logit model." *American Economic Review,* 105(1):272–98.

MCCALLUM, B. T. (2003): "The unique minimum state variable RE solution is E-stable in all well formulated linear models." NBER Working Papers 9960, National Bureau of Economic Research, Inc

MOLNAR, K. (2007): "Learning with expert advice." *Journal of the European Economic Association,* 5(2-3):420–432.

ORMENO, A. AND K. MOLNAR (2015): "Using survey data of inflation expectations in the estimation of learning and rational expectations models." *Journal of Money, Credit and Banking,* 47(4):673–699.

PFAJFAR, D. AND B. ŽAKELJ (2014): "Experimental evidence on inflation expectation formation." *Journal of Economic Dynamics and Control,* 44:147–168.

PFAJFAR, D. AND B. ŽAKELJ (2018): "Inflation expectations and monetary policy design: evidence from the laboratory." *Macroeconomic Dynamics,* 22(4):1035–1075.

SIMS, C. A. (2003): "Implications of rational inattention." *Journal of Monetary Economics,* 50 (3):665–690.

SLOBODYAN, S. AND R. WOUTERS (2012): "Learning in an estimated medium-scale DSGE model." *Journal of Economic Dynamics and Control,* 36(1):26–46.

SLOBODYAN, S., A. BOGOMOLOVA, AND D. KOLYUZHNOV (2016): "Stochastic gradient learning and instability: an example." *Macroeconomic Dynamics,* 20(03):777–790.

TIBSHIRANI, R. (1996): "Regression shrinkage and selection via the Lasso." *Journal of the Royal Statistical Society. Series B (Methodological),* 58(1):267–288.

WATSON, M. W. (2014): "Inflation persistence, the NAIRU, and the great recession." *American Economic Review,* 104(5):31–36.

ZOU, H. AND T. HASTIE (2005): "Regularization and variable selection via the Elastic Net." *Journal of the Royal Statistical Society, Series B,* 67:301–320.

Appendix A: Simple Model

A.1 Proposition 2.0.1

Proof. **Deriving RPE1.** The agents' PLM consistent with this RPE is

$$y_t = a_1 + b_1 w_t^1, \tag{A1}$$

therefore the ALM is given by

$$y_t = \alpha + \beta a_1 + \bar{b}_1 w_t^1 + \gamma_2 w_t^2 + \eta_t, \tag{A2}$$

with $\bar{b}_1 = \beta \rho_1 b_1 + \gamma_1$. In what follows, we will set $\alpha = 0$ and assume that the agents know this; therefore, $a_1 = 0$ as well.

In order for the agents to be using (A1) in equilibrium, it must be the case that b_1 is a coefficient in the regression of y_t on w_t^1, or

$$b_1 = \frac{Cov\left(y_t, w_t^1\right)}{Var\left(w_t^1\right)}. \tag{A3}$$

□

Computing the above expression for standardized variables we get

$$Cov\left(y_t, w_t^1\right) = E_t\left[\bar{b}_1 w_t^1 + \gamma_2 w_t^2 + \eta_t, w_t^1\right] = \bar{b}_1 + \gamma_2 \rho, \tag{A4}$$

$$b_1 = \bar{b}_1 + \gamma_2 \rho = \tag{A5}$$

$$= \beta \rho_1 b_1 + \gamma_1 + \gamma_2 \rho \Rightarrow b_1 = \frac{\gamma_1 + \gamma_2 \rho}{1 - \beta \rho_1}. \tag{A6}$$

Deriving RPE2. Similarly to the RPE1 case, we now have the PLM

$$y_t = a_2 + b_2 w_t^2, \tag{A7}$$

which implies that b_2 must be equal to the regression coefficient:

$$b_2 = \frac{Cov\left(y_t, w_t^2\right)}{Var\left(w_t^2\right)} = E_t\left[\bar{b}_1 w_t^1 + \gamma_2 w_t^2 + \eta_t, w_t^2\right] = \tag{A8}$$

$$= \left(\beta \rho_1 b_1 + \gamma_1\right) \rho + \gamma_2. \tag{A9}$$

E-stability. For the solution in (A6) and (A9) to be E-stable, the following should hold: $\frac{\partial T_{b_1}}{\partial b_1} < 1$ and $\frac{\partial T_{b_2}}{\partial b_2} < 1$. T_{b_1} is given by (A6) and T_{b_2} by (A9) . That is, $\frac{\partial T_{b_1}}{\partial b_1} = \frac{\partial[\beta \rho_1 b_1 + \gamma_1 + \gamma_2 \rho]}{\partial b_1} = \beta \rho_1$ and $\frac{\partial T_{b_2}}{\partial b_2} = \frac{\partial[(\beta \rho_1 b_1 + \gamma_1)\rho + \gamma_2]}{\partial b_2} = 0$. Thus, the only condition to be satisfied is:

$$\beta \rho_1 < 1. \tag{A10}$$

With both β<1 and $\rho_1 < 1$ both solutions are E-stable. Next, we want to ensure that the mean squared forecast error (MSFE) of the agents living in RPE1 and using (A1) is lower than the MSFE of the agents using (A7).

The forecast error and MSFE of the agents using (A1) are:

$$\begin{aligned} e_t^1 &= \bar{b}_1 w_t^1 + \gamma_2 w_t^2 + \eta_t - b_1 w_t^1, && \text{(A11)} \\ MSFE_1 &= E\left[e_t^1\right] = E\left[\left(\left(\bar{b}_1 - b_1\right) w_t^1 + \gamma_2 w_t^2 + \eta_t\right)^2\right]. && \text{(A12)} \end{aligned}$$

Similarly, for $MSFE_2$ we have the following expression

$$\begin{aligned} e_t^2 &= \bar{b}_1 w_t^1 + \gamma_2 w_t^2 + \eta_t - b_2 w_t^2, && \text{(A13)} \\ MSFE_2 &= E\left[e_t^2\right] = E\left[\left(\bar{b}_1 w_t^1 + \left(\gamma_2 - b_2\right) w_t^2 + \eta_t\right)^2\right]. && \text{(A14)} \end{aligned}$$

We are looking for the conditions under which $MSFE_1 < MSFE_2$:

$$\begin{aligned}
&: \ E\left[\left(\left(\bar{b}_1 - b_1\right) w_t^1 + \gamma_2 w_t^2 + \eta_t\right)^2\right] < E\left[\left(\bar{b}_1 w_t^1 + \left(\gamma_2 - b_2\right) w_t^2 + \eta_t\right)^2\right], \\
&: \ \left\{ \begin{array}{c} \left(\beta\rho_1 b_1 + \gamma_1 - b_1\right)^2 + \gamma_2^2 + \\ 2\gamma_2\left(\beta\rho_1 b_1 + \gamma_1 - b_1\right)\rho \end{array} \right\} < \left\{ \begin{array}{c} \left(\beta\rho_1 b_1 + \gamma_1\right)^2 + \left(\gamma_2 - b_2\right)^2 + \\ 2\left(\beta\rho_1 b_1 + \gamma_1\right)\left(\gamma_2 - b_2\right)\rho \end{array} \right\}, \\
&: \ \left\{ \begin{array}{c} \left(\beta\rho_1 b_1 + \gamma_1\right)^2 + b_1^2 \\ -2b_1\left(\beta\rho_1 b_1 + \gamma_1\right) + \gamma_2^2 \\ +2\gamma_2\left(\beta\rho_1 b_1 + \gamma_1\right)\rho - \\ -2\gamma_2 b_1\rho \end{array} \right\} < \left\{ \begin{array}{c} \left(\beta\rho_1 b_1 + \gamma_1\right)^2 + \gamma_2^2 \\ -2\gamma_2 b_2 + b_2^2 + \\ +2\gamma_2\left(\beta\rho_1 b_1 + \gamma_1\right)\rho - \\ -2b_2\left(\beta\rho_1 b_1 + \gamma_1\right)\rho \end{array} \right\}, \\
&: \ \left\{ \begin{array}{c} b_1^2 - 2b_1\left(\beta\rho_1 b_1 + \gamma_1\right) - \\ -2\gamma_2 b_1\rho \end{array} \right\} < \left\{ \begin{array}{c} -2\gamma_2 b_2 + b_2^2 - \\ -2b_2\left(\beta\rho_1 b_1 + \gamma_1\right)\rho\sigma_1\sigma_2 \end{array} \right\}, \\
&: \ \left\{ \begin{array}{c} b_1^2 - \\ -2b_1\underbrace{(\beta\rho_1 b_1 + \gamma_1 + \gamma_2\rho)}_{=b_1} \end{array} \right\} < \left\{ \begin{array}{c} b_2^2 - \\ -2b_2\underbrace{(\gamma_2 + \left(\beta\rho_1 b_1 + \gamma_1\right)\rho)}_{=b_2} \end{array} \right\}, \\
&: \ -b_1^2 < -b_2^2 \Rightarrow b_1^2 > b_2^2. && \text{(A15)}
\end{aligned}$$

The condition $b_1^2 > b_2^2$ has a very simple interpretation: in order for the MSFE of PLM1 to be lower than that of PLM2, the share of the variance of y_t explained by PLM1 must be higher than that of PLM2. Alternatively, the R^2 of regression (A1) must be higher than the R^2 of regression (A7).

Note that in the case of non-standardized variables, the condition will be re-written as $b_1^2\sigma_1^2 > b_2^2\sigma_2^2$, where the $b's$ will be related to the standardized case as $b_i^{nonscaled} = b_i^{scaled}\frac{\sigma y}{\sigma_i}$.

A.2 Analyzing the Conditions for $MSFE_1 < MSFE_2$

$$
\begin{aligned}
b_1^2 &> b_2^2 \Leftrightarrow b_1^2 > [(\beta\rho_1 b_1 + \gamma_1)\rho + \gamma_2]^2, \\
b_1^2 &> [(\beta\rho_1 b_1 + \gamma_1)\rho + \gamma_2]^2 = \left[\bar{b}_1\rho + \gamma_2\right]^2, \\
&: \left[\bar{b}_1 + \gamma_2\rho\right]^2 > \left[\bar{b}_1\rho + \gamma_2\right]^2, \\
&: \bar{b}_1^2 + 2\bar{b}_1\gamma_2\rho + (\gamma_2)^2\rho^2 > \bar{b}_1^2\rho^2 + 2\bar{b}_1\rho\gamma_2 + (\gamma_2)^2, \\
&: \bar{b}_1^2\left(1-\rho^2\right) > (\gamma_2)^2\left(1-\rho^2\right) \Rightarrow \bar{b}_1^2 > (\gamma_2)^2, \\
&: \left|\bar{b}_1\right\rangle \gamma_2.
\end{aligned} \tag{A16}
$$

Theoretically, there could be two separate cases:

$$
\begin{aligned}
&\textit{CASE I} : \bar{b}_1 > \gamma_2, &\text{(A17)}\\
&\textit{CASE II} : \bar{b}_1 < -\gamma_2. &\text{(A18)}
\end{aligned}
$$

As CASE I is the most obvious and will happen the most easily (assuming $\gamma_{1,2} > 0$, which is what we impose; otherwise, just re-define variable w_t^i so that $\gamma_{1,2}$ become positive), we start with this case.

A.2.1 CASE I: $\bar{b}_1 > \gamma_2$

Coming back to the condition of $MSFE_1 < MSFE_2$, we get

$$
\begin{aligned}
\bar{b}_1 &> \gamma_2, \\
\frac{\gamma_1 + \gamma_2\rho\beta\rho_1}{1-\beta\rho_1} &> \gamma_2, \\
\gamma_1 + \gamma_2\rho\beta\rho_1 &> \gamma_2 - \gamma_2\beta\rho_1, \\
\gamma_1 + \gamma_2\beta\rho_1(1+\rho) &> \gamma_2, \\
\frac{\gamma_1}{\gamma_2} + \beta\rho_1\rho + \beta\rho_1 - 1 &> 0.
\end{aligned} \tag{A19}
$$

Denoting the ratio of the coefficients on the observable shocks $\frac{\gamma_1}{\gamma_2}$ as Γ, we see that the condition for CASE I to be true is $\Gamma + \rho\beta\rho_1 > 1 - \beta\rho_1$, or

$$
\rho > \frac{1-\beta\rho_1-\Gamma}{\beta\rho_1}. \tag{A20}
$$

This condition is satisfied when $\rho_1 \to 1$ and Γ is large. Alternatively, when $\rho_1 \sim 0$ and Γ is small, so that the numerator is positive, this condition might amount to $\rho > 1$ and thus be impossible to satisfy.

A.2.2 CASE II: $\bar{b}_1 < -\gamma_2$

In this case, we have

$$
\begin{aligned}
\bar{b}_1 &< -\gamma_2, \\
\frac{\gamma_1 + \gamma_2\rho\beta\rho_1}{1-\beta\rho_1} &< -\gamma_2, \\
\gamma_1 + \gamma_2\rho\beta\rho_1 &< -\gamma_2 + \gamma_2\beta\rho_1, \\
\frac{\gamma_1}{\gamma_2} + \beta\rho_1\rho &< -1 + \beta\rho_1.
\end{aligned} \tag{A21}
$$

Using the notation just introduced, the condition $\bar{b}_1 < -\gamma_2$ amounts to $\Gamma + \rho\,\beta\rho_1 + (1-\beta\rho_1) < 0$. Consider its intersection with horizontal axis where $\Gamma = 0$. Then $\rho < -\frac{(1-\beta\rho_1)}{\beta\rho_1}$. As $|\rho| < 1$, the area where the solution exists is $-\frac{(1-\beta\rho_1)}{\beta\rho_1} > -1$, meaning that $\beta\rho_1 > 1/2$. That is, for the MSFE condition satisfied for $\bar{b}_1 < 0$, $\beta\rho_1$ must be larger than $1/2$.

To sum up, combining the two cases, we see that we need:

$$|\Gamma + \rho\,\beta\rho_1| > 1 - \beta\rho_1. \tag{A22}$$

A.3 Deriving the Sparse Weights

The sparse weights are derived by minimizing the following expression:

$$\min_{m\in[0,1]^n} \frac{1}{2} \sum_{i,j=1\ldots n} (1-m_i)\,\Lambda_{ij}\left(1-m_j\right) + \kappa \sum_{i,j=1\ldots n} m_i, \tag{A23}$$

with:

$$\begin{aligned}
\Lambda_{ij} &= -\sigma_{ij} a_{w_i} u_{aa} a_{w_j}, \\
a_{w_i} &= -u_{aa}^{-1} u_{aw_i}, \\
u_{aa} &= \frac{\partial^2 u_a}{\partial a^2} = \frac{\partial}{\partial a}\left(-(a - \bar{b}_1 w_t^1 - \bar{b}_2 w_t^2)\right) = -1, \\
u_{aw_i} &= \bar{b}_i.
\end{aligned}$$

Then the cost of inattention is

$$\Lambda_{ij} = \sigma_{ij}\bar{b}_i\bar{b}_j. \tag{A24}$$

Plugging the cost of inattention as in (A24) into (A23) we get the following problem:

$$\begin{aligned}
&\min_{m\in[0,1]^n} \frac{1}{2}\left\{(1-m_1)^2\,\bar{b}_1^2 + 2\,(1-m_1)\,(1-m_2)\,\sigma_{12}\bar{b}_1\bar{b}_2 + (1-m_2)^2\,\bar{b}_2^2\right\} + \\
&+\kappa\left(|m_1| + |m_2|\right).
\end{aligned} \tag{A25}$$

s.t.

$$\begin{aligned}
m_i &\leq 1, && \text{(A26)} \\
m_i &\geq 0, && \text{(A27)} \\
i &= 1,2. && \text{(A28)}
\end{aligned}$$

There are nine cases depending on which restriction is binding. Let us as start with the simplest case, with the inner solution for both weights: $0 < m_i < 1$.

1. The first-order conditions of (A25) with respect to m_1 and m_2 are:

$$\begin{aligned}
[m_1] &: \quad \kappa + \frac{1}{2}(-2\bar{b}_1^2(1-m_1) - 2\bar{b}_1\bar{b}_2\,(1-m_2)\,\sigma_{12} = 0, && \text{(A29)} \\
[m_2] &: \quad \kappa + \frac{1}{2}(-2\bar{b}_2^2(1-m_2) - 2\bar{b}_1\bar{b}_2\,(1-m_1)\,\sigma_{12} = 0. && \text{(A30)}
\end{aligned}$$

Solving (A29) and (A30) for m_1 and m_2 gives the expressions (14) and (15) in the text.

2. Consider the second case where $0 < m_2 < 1$, but $m_1 = 0$. The first-order conditions are then modified as:

$$[m_1] \quad : \quad \kappa + \frac{1}{2}(-2\bar{b}_1^2 - 2\bar{b}_1\bar{b}_2(1-m_2)\sigma_{12}) \geq 0, \tag{A31}$$

$$[m_2] \quad : \quad \kappa + \frac{1}{2}(-2\bar{b}_2^2(1-m_2) - 2\bar{b}_1\bar{b}_2\sigma_{12}) = 0, \tag{A32}$$

resulting in the following expressions:

$$[m_1] \quad : \quad m_2 \geq \frac{\bar{b}_1^2 - \kappa}{\bar{b}_1\bar{b}_2\sigma_{12}} + 1, \tag{A33}$$

$$[m_2] \quad : \quad m_2 = \frac{\bar{b}_1\bar{b}_2\sigma_{12} - \kappa}{\bar{b}_2^2} + 1. \tag{A34}$$

3. Consider the third case where $0 < m_1 < 1$, but $m_2 = 0$. The first-order conditions are then modified as:

$$[m_1] \quad : \quad \kappa + \frac{1}{2}(-2\bar{b}_1^2 + 2m_1\bar{b}_1^2 - 2\bar{b}_1\bar{b}_2\sigma_{12}) = 0, \tag{A35}$$

$$[m_2] \quad : \quad \kappa + \frac{1}{2}(-2\bar{b}_2^2 - 2\bar{b}_1\bar{b}_2\sigma_{12}(1-m_1)) \geq 0, \tag{A36}$$

resulting in the following expressions:

$$[m_1] \quad : \quad m_1 = \frac{\bar{b}_1\bar{b}_2\sigma_{12} - \kappa}{\bar{b}_1^2} + 1, \tag{A37}$$

$$[m_2] \quad : \quad \begin{cases} m_1 \geq \frac{\bar{b}_2^2 - \kappa}{\bar{b}_1\bar{b}_2\sigma_{12}} + 1 & \text{if } \bar{b}_1\bar{b}_2\sigma_{12} > 0, \\ m_1 \leq \frac{\bar{b}_2^2 - \kappa}{\bar{b}_1\bar{b}_2\sigma_{12}} + 1 & \text{if } \bar{b}_1\bar{b}_2\sigma_{12} < 0. \end{cases} \tag{A38}$$

Combining these two expressions yields:

$$\begin{cases} \frac{\bar{b}_1\bar{b}_2\sigma_{12} - \kappa}{\bar{b}_1^2} \geq \frac{\bar{b}_2^2 - \kappa}{\bar{b}_1\bar{b}_2\sigma_{12}} & \text{if } \bar{b}_1\bar{b}_2\sigma_{12} > 0, \\ \frac{\bar{b}_1\bar{b}_2\sigma_{12} - \kappa}{\bar{b}_1^2} \leq \frac{\bar{b}_2^2 - \kappa}{\bar{b}_1\bar{b}_2\sigma_{12}} & \text{if } \bar{b}_1\bar{b}_2\sigma_{12} < 0, \end{cases} \tag{A39}$$

$$\begin{cases} \bar{b}_1^2\bar{b}_2^2\sigma_{12}^2 - \kappa\bar{b}_1\bar{b}_2\sigma_{12} \geq \bar{b}_2^2\bar{b}_1^2 - \kappa\bar{b}_1^2 & \text{if } \bar{b}_1\bar{b}_2\sigma_{12} > 0, \\ \bar{b}_1^2\bar{b}_2^2\sigma_{12}^2 - \kappa\bar{b}_1\bar{b}_2\sigma_{12} \leq \bar{b}_2^2\bar{b}_1^2 - \kappa\bar{b}_1^2 & \text{if } \bar{b}_1\bar{b}_2\sigma_{12} < 0, \end{cases} \tag{A40}$$

$$\begin{cases} \bar{b}_1^2\bar{b}_2^2\sigma_{12}^2 - \bar{b}_2^2\bar{b}_1^2 \geq \kappa(\bar{b}_1\bar{b}_2\sigma_{12} - \bar{b}_1^2) & \text{if } \bar{b}_1\bar{b}_2\sigma_{12} > 0, \\ \bar{b}_1^2\bar{b}_2^2\sigma_{12}^2 - \bar{b}_2^2\bar{b}_1^2 \leq \kappa(\bar{b}_1\bar{b}_2\sigma_{12} - \bar{b}_1^2) & \text{if } \bar{b}_1\bar{b}_2\sigma_{12} < 0, \end{cases} \tag{A41}$$

As $\sigma_{12}^2 = \rho^2 < 1$ and $\bar{b}_1\bar{b}_2\sigma_{12} < \bar{b}_1^2$, the two inequalities establish the bound on κ: $\kappa \leq \frac{\bar{b}_1^2\bar{b}_2^2\sigma_{12}^2 - \bar{b}_2^2\bar{b}_1^2}{(\bar{b}_1\bar{b}_2\sigma_{12} - \bar{b}_1^2\sigma_1^2)}$
4. Consider the third case where both weights are zero. The first-order conditions are then modified as:

$$[m_1] \quad : \quad \kappa + \frac{1}{2}(-2\bar{b}_1^2 - 2\bar{b}_1\bar{b}_2\sigma_{12}) \geq 0, \tag{A42}$$

$$[m_2] \quad : \quad \kappa + \frac{1}{2}(-2\bar{b}_2^2 - 2\bar{b}_1\bar{b}_2\sigma_{12}) \geq 0, \tag{A43}$$

which give the conditions on attention cost, κ:

$$\kappa \geq \bar{b}_2^2 + \bar{b}_1\bar{b}_2\sigma_{12}, \tag{A44}$$
$$\kappa \geq \bar{b}_1^2 + \bar{b}_1\bar{b}_2\sigma_{12}. \tag{A45}$$

5. $m_1 = 1$ and $m_2 = 0$, with λ_1 as a Lagrange multiplier associated with (A26) for m_1:

$$[m_1] \quad : \quad \kappa - \bar{b}_1\bar{b}_2\sigma_{12} + \lambda_1 = 0, \tag{A46}$$
$$[m_2] \quad : \quad \kappa - \bar{b}_2^2 \geq 0, \tag{A47}$$

resulting in the conditions for κ: $\kappa \geq \bar{b}_2^2$ and $\kappa \leq \bar{b}_1\bar{b}_2\sigma_{12}$ (otherwise $\lambda_1 < 0$, which would violate the optimality conditions)

6. Similarly for $m_1 = 0$ and $m_2 = 1$, with λ_2 as a Lagrange multiplier associated with (A26) for m_2:

$$[m_1] \quad : \quad \kappa - \bar{b}_1^2 \geq 0, \tag{A48}$$
$$[m_2] \quad : \quad \kappa - \bar{b}_1\bar{b}_2\sigma_{12} + \lambda_2 = 0, \tag{A49}$$

resulting in the conditions for κ: $\kappa \geq \bar{b}_1^2$ and $\kappa \leq \bar{b}_1\bar{b}_2\sigma_{12}$ (otherwise $\lambda_2 < 0$, which would violate the optimality conditions)

7. For both weights equal to unity, $m_1 = 1$ and $m_2 = 1$

$$[m_1] \quad : \quad \kappa + \lambda_1 = 0, \tag{A50}$$
$$[m_2] \quad : \quad \kappa + \lambda_2 = 0, \tag{A51}$$

resulting in $\lambda_2 = \lambda_1 = \kappa = 0$,

8. For $m_1 = 1$ and $0 < m_2 < 1$:

$$[m_1] \quad : \quad \kappa - \bar{b}_1\bar{b}_2\,(1 - m_2)\,\sigma_{12} + \lambda_1 = 0, \tag{A52}$$
$$[m_2] \quad : \quad \kappa - \bar{b}_2^2(1 - m_2) = 0, \tag{A53}$$

resulting in:

$$m_2 = 1 - \frac{\kappa}{\bar{b}_2^2}, \tag{A54}$$
$$\kappa < \bar{b}_2^2, \tag{A55}$$
$$\bar{b}_1\bar{b}_2\sigma_{12} \geq \bar{b}_2^2, \tag{A56}$$

where the last inequality comes from the expression for $\lambda_1 = \kappa(\frac{\bar{b}_1\bar{b}_2\sigma_{12}}{\bar{b}_2^2} - 1)$ and the condition $\lambda_1 \geq 0$.

9. For $m_2 = 1$ and $0 < m_1 < 1$:

$$[m_1] \quad : \quad \kappa - \bar{b}_1^2(1 - m_1) = 0, \tag{A57}$$
$$[m_2] \quad : \quad \kappa - \bar{b}_1\bar{b}_2\,(1 - m_1)\,\sigma_{12} + \lambda_2 = 0, \tag{A58}$$

resulting in:

$$m_1 = 1 - \frac{\kappa}{\bar{b}_1^2}, \tag{A59}$$
$$\kappa < \bar{b}_1^2. \tag{A60}$$

Now define the value function as V_j, with j being the above solution case, and compare which of the solutions results in the minimum.

$$[0 < m_1 < 1, 0 < m_2 < 1] \qquad V_1 = 2\kappa - \frac{\kappa^2(\bar{b}_1^2 - 2\sigma_{12}\bar{b}_2\bar{b}_1 + \bar{b}_2^2)}{2(\bar{b}_2^2\bar{b}_1^2 - \sigma_{12}^2\bar{b}_1^2\bar{b}_2^2)}, \tag{A61}$$
$$[m_1 = 0, 0 < m_2 < 1] \qquad V_2 = \frac{1}{2}\bar{b}_1^2 + \kappa - \frac{(\sigma_{12}\bar{b}_1\bar{b}_2 - \kappa)^2}{2\bar{b}_2^2}, \tag{A62}$$
$$[0 < m_1 < 1, m_2 = 0] \qquad V_3 = \frac{1}{2}\bar{b}_2^2 + \kappa - \frac{(\sigma_{12}\bar{b}_1\bar{b}_2 - \kappa)^2}{2\bar{b}_1^2}, \tag{A63}$$
$$[m_1 = 0, m_2 = 0] \qquad V_4 = \frac{1}{2}(\bar{b}_2^2 + \bar{b}_1^2 + 2\sigma_{12}\bar{b}_1\bar{b}_2), \tag{A64}$$
$$[m_1 = 1, m_2 = 0] \qquad V_5 = \frac{1}{2}\bar{b}_2^2 + \kappa, \tag{A65}$$
$$[m_1 = 0, m_2 = 1] \qquad V_6 = \frac{1}{2}\bar{b}_1^2 + \kappa, \tag{A66}$$
$$[m_1 = 1, m_2 = 1] \qquad V_7 = 2\kappa, \tag{A67}$$
$$[m_1 = 1, 0 < m_2 < 1] \qquad V_8 = 2\kappa - \frac{\kappa^2}{2\bar{b}_2^2}, \tag{A68}$$
$$[0 < m_1 < 1, m_2 = 1] \qquad V_9 = 2\kappa - \frac{\kappa^2}{2\bar{b}_1^2}. \tag{A69}$$

Now compare the value functions.

$$V_8 < V_9 \quad : \quad 2\kappa - \frac{\kappa^2}{2\bar{b}_2^2} < 2\kappa - \frac{\kappa^2}{2\bar{b}_1^2}, \tag{A70}$$
$$\Rightarrow \quad \bar{b}_2^2 < \bar{b}_1^2. \tag{A71}$$

$$V_1 < V_3 \quad : \quad 2\kappa - \frac{\kappa^2(\bar{b}_1^2 - 2\rho\bar{b}_2\bar{b}_1 + \bar{b}_2^2)}{2\bar{b}_2^2\bar{b}_1^2(1-\rho^2)} < \frac{1}{2}\bar{b}_2^2 + \kappa - \frac{(\sigma_{12}\bar{b}_1\bar{b}_2 - \kappa)^2}{2\bar{b}_1^2}, \tag{A72}$$
$$: \quad \kappa(1 - \frac{\rho\bar{b}_2}{\bar{b}_1}) - \frac{\kappa^2(1 - 2\frac{\rho\bar{b}_2}{\bar{b}_1} + \rho^2\frac{\bar{b}_2^2}{\bar{b}_1^2})}{2\bar{b}_2^2(1-\rho^2)} < \frac{1}{2}\bar{b}_2^2(1-\rho^2), \tag{A73}$$
$$: \quad 2\kappa(1-\rho^2)(1 - \frac{\rho\bar{b}_2}{\bar{b}_1})\bar{b}_2^2 - \kappa^2(1 - \frac{\rho\bar{b}_2}{\bar{b}_1})^2 - \bar{b}_2^4(1-\rho^2)^2 < 0, \tag{A74}$$
$$: \quad -(\kappa(1 - \frac{\rho\bar{b}_2}{\bar{b}_1}) - \bar{b}_2(1-\rho^2))^2 < 0. \tag{A75}$$

$$V_1 < V_5 \quad : \quad 2\kappa - \frac{\kappa^2(\bar{b}_1^2 - 2\rho\bar{b}_2\bar{b}_1 + \bar{b}_2^2)}{2\bar{b}_2^2\bar{b}_1^2(1-\rho^2)} < \frac{1}{2}\bar{b}_2^2 + \kappa, \tag{A76}$$

$$: \quad 2\kappa\bar{b}_2^2 - \frac{\kappa^2(1 - 2\rho\frac{\bar{b}_2}{\bar{b}_1} + \frac{\bar{b}_2^2}{\bar{b}_1^2})}{(1-\rho^2)} < \bar{b}_2^4, \tag{A77}$$

$$: \quad -(\kappa - \bar{b}_2^2)^2 - \frac{\kappa^2(1 - 2\rho\frac{\bar{b}_2}{\bar{b}_1} + \frac{\bar{b}_2^2}{\bar{b}_1^2})}{(1-\rho^2)} + \kappa^2 < 0, \tag{A78}$$

$$: \quad -(\kappa - \bar{b}_2^2)^2 - \frac{\kappa^2(-2\rho\frac{\bar{b}_2}{\bar{b}_1} + \frac{\bar{b}_2^2}{\bar{b}_1^2} + \rho^2)}{(1-\rho^2)} < 0, \tag{A79}$$

$$: \quad -(\kappa - \bar{b}_2^2)^2 - \frac{\kappa^2(\rho - \frac{\bar{b}_2}{\bar{b}_1})^2}{(1-\rho^2)} < 0, \tag{A80}$$

$$\tag{A81}$$

$$V_1 < V_8 \quad : \quad 2\kappa - \frac{\kappa^2(\bar{b}_1^2 - 2\rho\bar{b}_2\bar{b}_1 + \bar{b}_2^2)}{2\bar{b}_2^2\bar{b}_1^2(1-\rho^2)} < 2\kappa - \frac{\kappa^2}{2\bar{b}_2^2}, \tag{A82}$$

$$: \quad -\frac{\kappa^2(\rho^2 - 2\rho\frac{\bar{b}_2}{\bar{b}_1} + \frac{\bar{b}_2^2}{\bar{b}_1^2})}{2\bar{b}_2^2(1-\rho^2)} < 0, \tag{A83}$$

$$: \quad -\frac{\kappa^2(\rho - \frac{\bar{b}_2}{\bar{b}_1})^2}{2\bar{b}_2^2(1-\rho^2)} < 0, \tag{A84}$$

$$V_3 < V_2 \quad : \quad \frac{1}{2}\bar{b}_2^2 + \kappa - \frac{(\sigma_{12}\bar{b}_1\bar{b}_2 - \kappa)^2}{2\bar{b}_1^2} < \frac{1}{2}\bar{b}_1^2 + \kappa - \frac{(\sigma_{12}\bar{b}_1\bar{b}_2 - \kappa)^2}{2\bar{b}_2^2}, \tag{A85}$$

$$\Rightarrow \quad \bar{b}_2^2 < \bar{b}_1^2. \tag{A86}$$

$$V_5 < V_6 \quad : \quad \frac{1}{2}\sigma_2^2\bar{b}_2^2 + \kappa < \frac{1}{2}\sigma_1^2\bar{b}_1^2 + \kappa, \tag{A87}$$

$$: \quad \Rightarrow \bar{b}_2^2 < \bar{b}_1^2. \tag{A88}$$

Thus, if an inner solution exists for $M_{\pi y}$, it outperforms the corner solution for M_π: $V_1 < V_5 < V6$, $V_1 < V_3 < V2$, and the corner solution for $M_{\pi y}$ with $w_1 = 1$: $V_1 < V_8 < V_9$.

A.4 Proof of Proposition 2.0.2

Proof. Proposition 2.0.2. To have $m_1 > m_2$ in the inner solution, we need:

$$\begin{aligned}
-\frac{\kappa}{\bar{b}_1\bar{b}_2\left(1-\rho^2\right)}\frac{\bar{b}_2 - \bar{b}_1\rho}{\bar{b}_1} &> -\frac{\kappa}{\bar{b}_1\bar{b}_2\left(1-\rho^2\right)}\frac{\bar{b}_1 - \rho\bar{b}_2}{\bar{b}_2}, \\
-\frac{1}{\bar{b}_1}\frac{\bar{b}_2 - \bar{b}_1\rho}{\bar{b}_1} &> -\frac{1}{\bar{b}_1}\frac{\bar{b}_1 - \rho\bar{b}_2}{\bar{b}_2}, \\
\left(\bar{b}_2 - \bar{b}_1\rho\right)\bar{b}_2 &< \bar{b}_1\left(\bar{b}_1 - \rho\bar{b}_2\right), \\
\bar{b}_2^2 - \bar{b}_1\bar{b}_2\rho &< \bar{b}_1^2 - \bar{b}_1\bar{b}_2\rho, \\
\bar{b}_2^2 &< \bar{b}_1^2.
\end{aligned} \tag{A89}$$

For κ such that the inner solution does not exist, solutions with $m_1 > m_2$ are optimal for $\bar{b}_2^2 < \bar{b}_1^2$ – see (A70)–(A87). □

A.5 Representing Inattention Cost as an Ellipse

Note that the first term in (A25) can be written as:

$$\frac{\bar{b}_1^2}{2}(\bar{x}^2 + 2\rho\, bs\, \bar{x}\bar{y} + \bar{y}^2 bs^2) = c, \tag{A90}$$

where c is a constant, $\bar{x} = 1 - m_1$, $\bar{y} = 1 - m_2$, and $bs = \frac{\bar{b}_2}{\bar{b}_1}$. The presence of the xy term indicates that the ellipse is rotated along the axes. The angle of rotation θ is then defined by $cot(2\theta) = \frac{1-bs^2}{2\rho\, bs}$.

Appendix B: Three-Equation New Keynesian Model

B.1 The Model

The model we use is similar to the textbook model as in Galí 2015, section 3, but with external habit formation. Here we sketch the main equations that reflect the difference from the textbook model. The detailed derivations can be found in Galí 2015, section 3. There is a continuum of identical households, each of them maximizing its intertemporal utility:

$$E_0 \sum_{t=0}^{\infty} \beta^t U_t, \tag{B1}$$

where β is the discount factor and U is the instantaneous utility function with external habit formation, such that:

$$U_t = e^{\tilde{g}_t}\left(\frac{1}{1-\sigma}(C_t - hC_{t-1})^{1-\sigma} - \frac{1}{1+\phi}L_t^{1+\phi}\right). \tag{B2}$$

Utility is increasing in consumption, C, relative to the previous period consumption, and is decreasing in labor supply, l. Habit persistence is governed by parameter h, σ is the coefficient of relative risk aversion, and ϕ is the inverse of the elasticity of work effort with respect to the real wage. $e^{\tilde{g}_t}$ is the shock to the discount rate.

Households maximize utility subject to the budget constraint:

$$P_t C_t + Q_t B_t \leq B_{t-1} + W_t L_t + D_t, \tag{B3}$$

where P_t is the price of the consumption good, W_t is the nominal wage, B_t is the quantity of one-period nominally riskless discount bonds, each of them paying Q_t on maturity, and D_t are the dividends that households receive as firm owners. The problem results in the following optimality conditions for consumption for a representative household:

$$Q_t = \beta E_t\left(\left(\frac{e^{\tilde{g}_{t+1}}(C_{t+1} - hC_t)}{e^{\tilde{g}_t}(C_t - hC_{t-1})}\right)^{-\sigma}(-h)\frac{P_t}{P_{t+1}}\right). \tag{B4}$$

After linearization, the consumption equation is:

$$\hat{c}_t = \frac{h}{1+h}\hat{c}_{t-1} + \frac{1}{1+h}E_t\hat{c}_{t+1} - \frac{1-h}{(1+h)\sigma}(i_t - E_t\pi_{t+1}) + \frac{1-h}{(1+h)\sigma}(\tilde{g}_t - E_t\tilde{g}_{t+1}). \tag{B5}$$

In this simple model environment with $y_t = c_t$, the linearized equation for the output gap is:

$$\hat{y}_t = \frac{h}{1+h}\hat{y}_{t-1} + \frac{1}{1+h}E_t\hat{y}_{t+1} - \frac{1-h}{(1+h)\sigma}(i_t - E_t\pi_{t+1}) + \frac{1-h}{(1+h)\sigma}\tilde{g}_t, \quad \text{(B6)}$$

where we have used the fact that $E_t\tilde{g}_{t+1} = 0$. Under extreme habit formation with $\lim_{h\to\infty}$, the equation for the output gap loses the forward-looking argument:

$$\hat{y}_t = \hat{y}_{t-1} - \frac{1}{\sigma}(i_t - E_t\pi_{t+1}) + g_t, \quad \text{(B7)}$$

with rescaled $g_t = \frac{1}{\sigma}\tilde{g}_t$. This is equation (17) in the text.

As the derivation of the Phillips curve, (18), is identical to Galí 2015, section 3, the reader is referred to the textbook for the details.

B.2 Rational Expectations Equilibria

Proof. Proposition 3.0.1

Deriving REE with the forward-looking policy rule. The solution of the model under the forward-looking rule is:

$$y_t = \frac{1-\phi\pi}{\sigma}E_t\pi_{t+1} + y_{t-1} + g_t, \quad \text{(B8)}$$

$$\pi_t = (\beta + \omega\frac{1-\phi\pi}{\sigma})E_t\pi_{t+1} + \omega y_{t-1} + \omega g_t + u_t. \quad \text{(B9)}$$

We re-define the coefficients in (B8) and (B9) as:

$$b_y = \frac{1-\phi\pi}{\sigma}, \quad \text{(B10)}$$

$$b_\pi = \beta + \omega\frac{1-\phi\pi}{\sigma}. \quad \text{(B11)}$$

$$c_\pi = \omega. \quad \text{(B12)}$$

There are two sets of solutions consistent with rational expectations. One set is associated with the following forecasting rule:

$$\pi_t^{REE} = cy_{t-1} + b\pi_{t-1} + dg_t + zu_t, \quad \text{(B13)}$$

$$y_t^{REE} = \tilde{c}y_{t-1} + \tilde{b}\pi_{t-1} + \tilde{d}g_t + \tilde{z}u_t, \quad \text{(B14)}$$

$$E_t\pi_{t+1}^{REE} = cy_t + b\pi_t = c(\tilde{c}+b)y_{t-1} + (c\tilde{b}+b^2)\pi_{t-1} + (c\tilde{d}+bd)g_t + (bz+c\tilde{z})u_t. \quad \text{(B15)}$$

Plugging the forecasting rules into (B9) and using the method of undetermined coefficients yields:

$$b = b_\pi(c\tilde{b}+b^2), \quad \text{(B16)}$$

$$c = b_\pi c(\tilde{c}+b) + \omega, \quad \text{(B17)}$$

$$d = b_\pi(c\tilde{d}+bd) + \omega, \quad \text{(B18)}$$

$$z = b_\pi(bz+c\tilde{z}) + 1, \quad \text{(B19)}$$

$$\tilde{b} = b_y(c\tilde{b}+b^2), \quad \text{(B20)}$$

$$\tilde{c} = b_y c(\tilde{c}+b) + 1, \quad \text{(B21)}$$

$$\tilde{z} = b_y(bz+c\tilde{z}). \quad \text{(B22)}$$

With the solution:

$$\begin{aligned} b &= \frac{1}{\beta}, & \text{(B23)} \\ c &= -\frac{\omega}{\beta}, & \text{(B24)} \\ 0 &= 0, & \text{(B25)} \\ 0 &= \beta, & \text{(B26)} \\ \tilde{b} &= -\frac{b_y}{\omega b_\pi}, & \text{(B27)} \\ \tilde{c} &= \frac{\beta - \omega b_y}{\beta b_\pi}, & \text{(B28)} \\ 0 &= 0, & \text{(B29)} \\ \tilde{z} &\in \emptyset. & \text{(B30)} \end{aligned}$$

Clearly, the REE solution is not unique, as there are multiple solutions for d. Note that there is no solution for z.

Another set of REE-consistent solutions is the MSV solution of the form:

$$\begin{aligned} \pi_t^{MSV} &= cy_{t-1} + dg_t + zu_t, & \text{(B31)} \\ E_t\pi_{t+1}^{MSV} &= cy_t = cb_yE_t\pi_{t+1} + cy_{t-1} + cg_t, & \text{(B32)} \\ E_t\pi_{t+1}^{MSV} &= \frac{c}{1-cb_y}y_{t-1} + \frac{c}{1-cb_y}g_t. & \text{(B33)} \end{aligned}$$

Plugging the forecasting rule into (B9) yields:

$$\begin{aligned} \pi_t &= \left(\frac{cb_\pi}{1-cb_y} + c_\pi\right)y_{t-1} + \left(\frac{cb_\pi}{1-cb_y} + c_\pi\right)g_t + u_t, & \text{(B34)} \\ y_t &= \frac{1}{1-cb_y}y_{t-1} + \frac{1}{1-cb_y}g_t, & \text{(B35)} \end{aligned}$$

with the coefficients defined as in (B10) and (B12). Using the method of undetermined coefficients, we solve for the forecasting rule coefficients.

$$\begin{aligned} c &= \frac{cb_\pi}{1-cb_y} + c_\pi, & \text{(B36)} \\ d &= \frac{cb_\pi}{1-cb_y} + c_\pi, & \text{(B37)} \\ z &= 1. & \text{(B38)} \end{aligned}$$

The MSV solution is:

$$\begin{aligned} d &= \frac{cb_\pi}{1-cb_y} + c_\pi, & \text{(B39)} \\ z &= 1, & \text{(B40)} \\ c_2 &= \frac{1-\beta+\sqrt{(1-\beta)^2 - 4\omega\frac{1-\phi_\pi}{\sigma}}}{2\frac{1-\phi_\pi}{\sigma}}, & \text{(B41)} \\ c_1 &= \frac{1-\beta-\sqrt{(1-\beta)^2 - 4\omega\frac{1-\phi_\pi}{\sigma}}}{2\frac{1-\phi_\pi}{\sigma}}. & \text{(B42)} \end{aligned}$$

Analyzing the solution in (B42), note that the solution exists only if $\phi_\pi > 1 - \frac{(1-\beta)^2\sigma}{4\omega}$. In the spirit of McCallum (2003), we consider a MSV solution that in the special case of $\omega = 0$ would converge to 0. Clearly, this is (B42). The ALM coefficients are then:

$$\bar{c}_y^{MSV} = \frac{1}{1-b_y c}, \tag{B43}$$

$$\bar{c}_\pi^{MSV} = \frac{cb_\pi}{1-b_y c} + c_\pi. \tag{B44}$$

(B45)

E-stability. For the MSV solution to be E-stable, the following should hold

$$eig\begin{pmatrix} \frac{\partial T_c}{\partial c} & \frac{\partial T_c}{\partial d} \\ \frac{\partial T_{d,c}}{\partial c} & \frac{\partial T_{d,c}}{\partial d} \end{pmatrix} = eig\begin{pmatrix} \frac{b_\pi}{(1-b_y c)^2} & 0 \\ \frac{b_\pi}{(1-b_y c)^2} & 0 \end{pmatrix} < 1.$$

The eigenvalues of the above matrix are $[0, \frac{b_\pi}{(1-b_y c)^2}]$. The criterion for E-stability is then:

$$\frac{b_\pi}{(1-b_y c)^2} < 1 \Rightarrow 1 - b_\pi - 2b_y c + (b_y c)^2 > 0. \tag{B46}$$

Note that the above inequality is an upward open parabola in c. That is, if c_1 and c_2 are the roots of the parabola, and the roots exist, the inequality is satisfied when c lies outside the interval $[c_1, c_2]$. If roots do not exist, then the inequality is satisfied on the whole parameter range. The roots of the parabola are $c_{1,2} = \frac{1 \pm \sqrt{b_\pi}}{b_y}$. The real roots do not exist when $b_\pi < 0$, that is, $\phi_\pi > 1 + \frac{\beta\sigma}{\omega}$.

If $\phi_\pi < 1 + \frac{\beta\sigma}{\omega}$ and $\phi_\pi > 1$, then:

$$c_1 = \frac{1 + \sqrt{\beta + \frac{\omega(1-\phi_\pi)}{\sigma}}}{\frac{(1-\phi_\pi)}{\sigma}}, \tag{B47}$$

$$c_2 = \frac{1 - \sqrt{\beta + \frac{\omega(1-\phi_\pi)}{\sigma}}}{\frac{(1-\phi_\pi)}{\sigma}}, \tag{B48}$$

with $c_2 > c_1$ as the denominator is negative. It is straightforward to show that c in (B42) is larger than c_2:

$$\frac{1 - \beta - \sqrt{(1-\beta)^2 - 4\omega\frac{1-\phi_\pi}{\sigma}}}{2\frac{1-\phi_\pi}{\sigma}} > \frac{1 - \sqrt{\beta + \frac{\omega(1-\phi_\pi)}{\sigma}}}{\frac{(1-\phi_\pi)}{\sigma}}, \tag{B49}$$

$$\Rightarrow \quad \frac{1 - \beta - \sqrt{(1-\beta)^2 - 4\omega\frac{1-\phi_\pi}{\sigma}}}{2} < 1 - \sqrt{\beta + \frac{\omega(1-\phi_\pi)}{\sigma}}. \tag{B50}$$

Now, if $\omega = 0$, the above inequality transforms into $0 < 1 - \sqrt{\beta}$, with $\beta < 1$. When ω increases, the left-hand side decreases, but the right hand side increases. That is why the inequality is satisfied.

If $\phi_\pi < 1 + \frac{\beta\sigma}{\omega}$ and $\phi_\pi < 1$, then $c_2 < c_1$. Clearly, $c < c_1$. For the solution to be E-stable, it must then hold that $c < c_2$, which is equivalent to:

$$\frac{1-\beta-\sqrt{(1-\beta)^2-4\omega\frac{1-\phi_\pi}{\sigma}}}{2} < 1-\sqrt{\beta+\frac{\omega(1-\phi_\pi)}{\sigma}}, \tag{B51}$$

$$\Rightarrow \quad -(b-a)-\sqrt{b^2-a^2} < 0, \tag{B52}$$

where $b = \frac{1+\beta}{2}$ and $a = \sqrt{b\pi}$. The inequality holds for $b-a>0$, that is for $\phi_\pi > 1-\frac{(1-\beta)^2\sigma}{4\omega}$.

Determinacy. Representing the ALM as $z_t = A + Bz_{t-1} + U_t$, with $z_t = (y_t, \pi_t)'$, A and U_t are vectors of the constants and shocks respectively, and

$$B = \begin{pmatrix} \bar{c}_y^{MSV} & \bar{b}_y^{MSV} \\ \bar{c}_\pi^{MSV} & \bar{b}_\pi^{MSV} \end{pmatrix}. \tag{B53}$$

For the solution to be determinate, both eigenvalues of B53 must be inside the unit circle, where the eigenvalues are $(0, \bar{c}_y^{MSV})$. That is, the condition for determinacy is:

$$|\bar{c}_y^{MSV}| < 1 \rightarrow |\frac{1}{1-b_y c}| < 1 \rightarrow b_y c < 0 \rightarrow b_y < 0 \rightarrow \phi_\pi > 1. \tag{B54}$$

Where in the derivations we have used $c > 0$ and $c < 1$, that follows from (B50). Thus, with $\phi_\pi > 1$, the MSV solution is both determinate and E-Stable.

Deriving REE when the rule with contemporaneous inflation expectations is used. The model solution is:

$$y_t = -\frac{\phi_\pi}{\sigma}E_t\pi_t + \frac{1}{\sigma}E_t\pi_{t+1} + y_{t-1} + g_t, \tag{B55}$$

$$\pi_t = (\beta+\frac{\omega}{\sigma})E_t\pi_{t+1} - \frac{\omega\phi_\pi}{\sigma}E_t\pi_t + \omega y_{t-1} + \omega g_t + u_t. \tag{B56}$$

The forecasting rule consistent with REE will be:

$$\pi_t^{REE} = cy_{t-1} + b\pi_{t-1} + dg_t + zu_t, \tag{B57}$$

$$y_t^{REE} = \tilde{c}y_{t-1} + \tilde{b}\pi_{t-1} + \tilde{d}g_t + \tilde{z}u_t, \tag{B58}$$

$$E_t\pi_{t+1}^{REE} = cy_t + b\pi_t = c(\tilde{c}+b)y_{t-1} + (c\tilde{b}+b^2)\pi_{t-1} + (c\tilde{d}+bd)g_t + (bz+c\tilde{z})u_t. \tag{B59}$$

Plugging the forecasting rule into (B9) and using the method of undetermined coefficients,we get the following mapping:

$$\begin{aligned}
b &= \left(\beta+\frac{\omega}{\sigma}\right)(c\tilde{b}+b^2)-\frac{\omega}{\sigma}\phi_\pi b, && \text{(B60)}\\
c &= \left(\beta+\frac{\omega}{\sigma}\right)c(\tilde{c}+b)-\frac{\omega}{\sigma}\phi_\pi c+\omega, && \text{(B61)}\\
d &= \left(\beta+\frac{\omega}{\sigma}\right)(c\tilde{d}+bd)-\frac{\omega}{\sigma}\phi_\pi d+\omega, && \text{(B62)}\\
z &= \left(\beta+\frac{\omega}{\sigma}\right)(c\tilde{z}+bz)-\frac{\omega}{\sigma}\phi_\pi z+1, && \text{(B63)}\\
\tilde{b} &= -\frac{\omega}{\sigma}\phi_\pi b+\frac{1}{\sigma}(c\tilde{b}+b^2), && \text{(B64)}\\
\tilde{c} &= -\frac{\omega}{\sigma}\phi_\pi c+\frac{c}{\sigma}(\tilde{c}+b), && \text{(B65)}\\
\tilde{d} &= -\frac{\omega}{\sigma}\phi_\pi d+\frac{1}{\sigma}(c\tilde{d}+bd), && \text{(B66)}\\
\tilde{z} &= -\frac{\omega}{\sigma}\phi_\pi z+\frac{1}{\sigma}(c\tilde{z}+bz), && \text{(B67)}\\
& && \text{(B68)}
\end{aligned}$$

And the set of solutions compatible with rational expectations is:

$$\begin{aligned}
b &= \frac{1}{\beta}, && \text{(B69)}\\
c &= -\frac{\omega}{\beta}, && \text{(B70)}\\
0 &= 0, && \text{(B71)}\\
z &= \emptyset. && \text{(B72)}
\end{aligned}$$

Again, there are multiple solutions for d, and there is no solution for z.

Let us now consider the MSV solution. With the MSV forecasting rules as in (B31):

$$\begin{aligned}
\pi_t^{MSV} &= cy_{t-1}+dg_t+zu_t, && \text{(B73)}\\
\pi_{t+1}^{MSV} &= cy_t=-\frac{\phi_\pi c}{\sigma}E_t\pi_t+\frac{c}{\sigma}E_t\pi_{t+1}+cy_{t-1}+cg_t, && \text{(B74)}\\
\pi_{t+1}^{MSV} &= -\frac{\sigma}{\sigma-c}\frac{\phi_\pi c}{\sigma}(cy_{t-1}+dg_t+zu_t)+\frac{\sigma}{\sigma-c}cy_{t-1}+\frac{\sigma}{\sigma-c}cg_t\\
&= \left(-\frac{\sigma}{\sigma-c}\frac{\phi_\pi c^2}{\sigma}+\frac{\sigma}{\sigma-c}c\right)y_{t-1}+\left(-\frac{\sigma}{\sigma-c}\frac{\phi_\pi cd}{\sigma}+\frac{\sigma}{\sigma-c}c\right)g_t-\frac{\sigma z}{\sigma-c}\frac{\phi_\pi c}{\sigma}u_t. && \text{(B75)}
\end{aligned}$$

Plugging the expressions into the rule of motion for inflation:

$$\begin{aligned}
\pi_t &= \left(\beta+\frac{\omega}{\sigma}\right)\left(\left(-\frac{\sigma}{\sigma-c}\frac{\phi_\pi c^2}{\sigma}+\frac{\sigma}{\sigma-c}c\right)y_{t-1}+\left(-\frac{\sigma}{\sigma-c}\frac{\phi_\pi cd}{\sigma}+\frac{\sigma}{\sigma-c}c\right)g_t-\frac{\sigma z}{\sigma-c}\frac{\phi_\pi c}{\sigma}u_t\right)\\
&- \frac{\omega\phi_\pi}{\sigma}(cy_{t-1}+dg_t+zu_t)+\omega y_{t-1}+\omega g_t+u_t\\
&= \left((\beta+\frac{\omega}{\sigma})(-\frac{\sigma}{\sigma-c}\frac{\phi_\pi c^2}{\sigma}+\frac{\sigma}{\sigma-c}c)-\frac{\omega\phi_\pi}{\sigma}c+\omega\right)y_{t-1}\\
&+ \left((\beta+\frac{\omega}{\sigma})(-\frac{\sigma}{\sigma-c}\frac{\phi_\pi cd}{\sigma}+\frac{\sigma}{\sigma-c}c)-\frac{\omega\phi_\pi}{\sigma}d+\omega\right)g_t+\left(-\frac{\omega\phi_\pi z}{\sigma}+1\right)u_t. && \text{(B76)}
\end{aligned}$$

For output:

$$
\begin{aligned}
y_t &= \left(-\frac{\phi_\pi}{\sigma}c - \frac{\phi_\pi c^2}{\sigma(\sigma-c)} + \frac{c}{\sigma-c} + 1\right) y_{t-1} - \frac{\phi_\pi}{\sigma} d g_t - \frac{\phi_\pi}{\sigma} z u_t + \\
&\left(-\frac{\frac{\phi_\pi c d}{\sigma-c}}{\sigma} + \frac{c}{\sigma-c}\right) g_t - \frac{z\phi_\pi c}{\sigma(\sigma-c)} u_t + g_t, \qquad \text{(B77)}
\end{aligned}
$$

$$
\begin{aligned}
c &= \left(\frac{\beta\sigma+\omega}{\sigma}\right)\left(-\frac{1}{\sigma-c}\frac{\phi_\pi c^2}{1} + \frac{\sigma}{\sigma-c}c\right) - \frac{\omega\phi_\pi}{\sigma}c + \omega \\
&= -\frac{\beta\sigma+\omega}{\sigma-c}\frac{\phi_\pi c^2}{\sigma} + \frac{\beta\sigma+\omega}{\sigma-c}c - \frac{\omega\phi_\pi}{\sigma}c + \omega, \qquad \text{(B78)} \\
&\sigma c - c^2 = -\beta\sigma\frac{\phi_\pi c^2}{\sigma} - \omega\frac{\phi_\pi c^2}{\sigma} + \beta\sigma c + \omega c - \omega c\frac{\phi_\pi}{\sigma}\sigma + \omega c^2\frac{\phi_\pi}{\sigma} - \omega c + \omega\sigma \\
&c\left(\sigma - \beta\sigma + \omega\phi_\pi\right) + c^2\left(-1 + \beta\phi_\pi\right) = \omega\sigma.
\end{aligned}
$$

$$
\begin{aligned}
d &= \left(\beta + \frac{\omega}{\sigma}\right)\left(-\frac{\sigma}{\sigma-c}\frac{\phi_\pi c d}{\sigma} + \frac{\sigma}{\sigma-c}c\right) - \frac{\omega\phi_\pi}{\sigma}d + \omega \\
&= \left(\frac{\beta\sigma+\omega}{\sigma-c}\right) c \left(1 - \frac{\phi_\pi d}{\sigma}\right) - \frac{\omega\phi_\pi}{\sigma}d, \qquad \text{(B79)} \\
z &= -\frac{\omega\phi_\pi z}{\sigma} + 1. \qquad \text{(B80)}
\end{aligned}
$$

$$
c = \frac{(1-\beta) + \frac{\phi_\pi}{\sigma}\omega \pm \sqrt{\left((1-\beta) + \frac{\phi_\pi}{\sigma}\omega\right)^2 - 4\omega\left(\frac{(1-\beta\phi_\pi)}{\sigma}\right)}}{2\frac{(1-\beta\phi_\pi)}{\sigma}}. \qquad \text{(B81)}
$$

Using the same logic as above, we leave only the MSV solution that is zero if $\omega = 0$. That is:

$$
c = \frac{(1-\beta) + \frac{\phi_\pi}{\sigma}\omega - \sqrt{\left((1-\beta) + \frac{\phi_\pi}{\sigma}\omega\right)^2 - 4\omega\left(\frac{(1-\beta\phi_\pi)}{\sigma}\right)}}{2\frac{(1-\beta\phi_\pi)}{\sigma}}. \qquad \text{(B82)}
$$

Note that $c \geq 0$. It can be written as:

$$
\begin{aligned}
c &= \frac{a - \sqrt{a^2 - 4\omega b}}{2b}, \\
a &= (1-\beta) + \frac{\phi_\pi}{\sigma}\omega, \\
b &= \frac{(1-\beta\phi_\pi)}{\sigma}.
\end{aligned}
$$

Now, if $b < 0$, both the numerator and the denominator in c are negative, making c positive. If $b > 0$, both the numerator and the denominator are positive, and c is positive. It is instructive for future derivations to show that $c < \sigma$ for $\phi_\pi > 1$. Suppose that $c > \sigma$:

$$
\begin{aligned}
c &= \frac{a - \sqrt{a^2 - 4\omega b}}{2b} > \sigma, \Longrightarrow \begin{cases} a - \sqrt{a^2 - 4\omega b} > 2b\sigma, & \text{if } b > 0, \\ a - \sqrt{a^2 - 4\omega b} < 2b\sigma, & \text{if } b < 0, \end{cases} \\
&\Longrightarrow \begin{cases} a - 2b\sigma > \sqrt{a^2 - 4\omega b}, & \text{if } b > 0, \\ a - 2b\sigma < \sqrt{a^2 - 4\omega b}, & \text{if } b < 0, \end{cases} \Longrightarrow \begin{cases} a^2 + 4b^2\sigma^2 - 4ba\sigma > a^2 - 4\omega b, & \text{if } b > 0, \\ a^2 + 4b^2\sigma^2 - 4ba\sigma < a^2 - 4\omega b, & \text{if } b < 0, \end{cases} \\
&\Longrightarrow \begin{cases} b\sigma^2 - a\sigma > -\omega, & \text{if } b > 0, \\ b\sigma^2 - a\sigma > -\omega, & \text{if } b < 0, \end{cases} \\
&\Longrightarrow \begin{cases} b > \frac{a}{\sigma} - \frac{\omega}{\sigma^2}, & \text{if } b > 0, \\ b > \frac{a}{\sigma} - \frac{\omega}{\sigma^2}, & \text{if } b < 0, \end{cases} \Longrightarrow \begin{cases} \frac{(1-\beta\phi_\pi)}{\sigma} > \frac{(1-\beta)+\frac{\phi_\pi\omega}{\sigma}}{\sigma} - \frac{\omega}{\sigma^2}, & \text{if } b > 0, \\ \frac{(1-\beta\phi_\pi)}{\sigma} < \frac{(1-\beta)+\frac{\phi_\pi\omega}{\sigma}}{\sigma} - \frac{\omega}{\sigma^2}, & \text{if } b > 0, \end{cases} \\
&\Longrightarrow \begin{cases} (1-\beta\phi_\pi) > (1-\beta) + \frac{\phi_\pi\omega}{\sigma} - \frac{\omega}{\sigma}, & \text{if } b > 0, \\ (1-\beta\phi_\pi) > (1-\beta) + \frac{\phi_\pi\omega}{\sigma} - \frac{\omega}{\sigma}, & \text{if } b < 0, \end{cases} \Longrightarrow \begin{cases} (1-\phi_\pi)(\beta + \frac{\omega}{\sigma}) > 0, & \text{if } b > 0, \\ (1-\phi_\pi)(\beta + \frac{\omega}{\sigma}) > 0, & \text{if } b < 0. \end{cases}
\end{aligned}
$$

Clearly, $c > \sigma$ only if $\phi_\pi < 1$.

E-stability. We calculate the eigenvalues of the following matrix:

$$
\begin{pmatrix} \frac{(\beta\sigma+\omega)(\frac{c^2\phi_\pi}{\sigma} - 2c\phi_\pi + \sigma)}{(\sigma-c)^2} - \frac{\omega\phi_\pi}{\sigma} & 0 & 0 \\ \frac{(\beta\sigma+\omega)\sigma(1-\phi_\pi d)}{(\sigma-c)^2} & -\frac{(\beta\sigma+\omega)}{\sigma-c}\frac{c\phi_\pi}{\sigma} - \frac{\omega\phi_\pi}{\sigma} & 0 \\ 0 & 0 & -\frac{\omega\phi_\pi}{\sigma} \end{pmatrix}.
$$

The eigenvalues are $[-\frac{\omega\phi_\pi}{\sigma}, \frac{(\beta\sigma+\omega)(\frac{c^2\phi_\pi}{\sigma} - 2c\phi_\pi+\sigma)}{(\sigma-c)^2} - \frac{\omega\phi_\pi}{\sigma}, -\frac{(\beta\sigma+\omega)}{\sigma-c}\frac{c\phi_\pi}{\sigma} - \frac{\omega\phi_\pi}{\sigma}]$. Clearly, $-\frac{\omega\phi_\pi}{\sigma} < 0$, $-\frac{(\beta\sigma+\omega)}{\sigma-c}\frac{c\phi_\pi}{\sigma} - \frac{\omega\phi_\pi}{\sigma} < 1$ for $c < \sigma$:

$$
\begin{aligned}
& -\frac{\beta\sigma+\omega}{\sigma-c}\frac{c\phi_\pi}{\sigma} - \frac{\omega\phi_\pi}{\sigma} = \frac{\phi_\pi}{\sigma}\frac{-\beta\sigma c - \omega c - \omega\sigma + c\omega}{\sigma - c} \\
&= \frac{\phi_\pi}{\sigma - c}(-\beta c - \omega) < 1, \Longrightarrow \begin{cases} \phi_\pi(-\beta c - \omega) < \sigma - c, & \text{if } \sigma - c > 0, \\ \phi_\pi(-\beta c - \omega) > \sigma - c, & \text{if } \sigma - c < 0, \end{cases} \\
&\Longrightarrow \begin{cases} c\frac{1-\beta\phi_\pi}{\sigma} < 1 + \frac{\phi_\pi\omega}{\sigma}, & \text{if } \sigma - c > 0, \\ c\frac{1-\beta\phi_\pi}{\sigma} > 1 + \frac{\phi_\pi\omega}{\sigma}, & \text{if } \sigma - c < 0, \end{cases} \\
&\Longrightarrow \begin{cases} \begin{cases} c < \frac{1+\frac{\phi_\pi\omega}{\sigma}}{\frac{1-\beta\phi_\pi}{\sigma}}, & \text{if } \sigma - c > 0 \text{ and } 1 - \beta\phi_\pi > 0, \\ c > \frac{1+\frac{\phi_\pi\omega}{\sigma}}{\frac{1-\beta\phi_\pi}{\sigma}}, & \text{if } \sigma - c > 0 \text{ and } 1 - \beta\phi_\pi < 0, \end{cases} \\ \begin{cases} c > \frac{1+\frac{\phi_\pi\omega}{\sigma}}{\frac{1-\beta\phi_\pi}{\sigma}}, & \text{if } \sigma - c < 0 \text{ and } 1 - \beta\phi_\pi > 0, \\ c < \frac{1+\frac{\phi_\pi\omega}{\sigma}}{\frac{1-\beta\phi_\pi}{\sigma}}, & \text{if } \sigma - c < 0 \text{ and } 1 - \beta\phi_\pi < 0. \end{cases} \end{cases}
\end{aligned}
$$

Note that in (B82), $1 - \beta + \frac{\phi_\pi\omega}{\sigma} < 1 + \frac{\phi_\pi\omega}{\sigma}$. Also, for $1 - \beta\phi_\pi < 0$, $c > \frac{1+\frac{\phi_\pi\omega}{\sigma}}{\frac{1-\beta\phi_\pi}{\sigma}}$ and is smaller otherwise. Then, all cases with $\sigma - c < 0$ belong to the empty set. Thus, for $c < \sigma$, the last eigenvalue is smaller than unity. For the second eigenvalue to be smaller than unity:

$$\begin{aligned}
&\frac{(\beta\sigma+\omega)(\frac{c^2\phi_\pi}{\sigma}-2c\phi_\pi+\sigma)}{(\sigma-c)^2}<1+\frac{\omega\phi_\pi}{\sigma},\\
&\Longrightarrow (\beta\sigma+\omega)(\frac{c^2\phi_\pi}{\sigma}-2c\phi_\pi+\sigma)<(\sigma-c)^2(1+\frac{\omega\phi_\pi}{\sigma}), \Longrightarrow\\
&\beta\sigma\frac{c^2\phi_\pi}{\sigma}+\omega\frac{c^2\phi_\pi}{\sigma}-2\beta\sigma c\phi_\pi-2c\phi_\pi\omega+\sigma\beta\sigma+\sigma\omega<(\sigma-c)^2+\sigma^2\frac{\omega\phi_\pi}{\sigma}-2\sigma c\frac{\omega\phi_\pi}{\sigma}+c^2\frac{\omega\phi_\pi}{\sigma},\\
&\Longrightarrow \phi_\pi(\beta c^2-2\beta\sigma c-\sigma\omega)<(\sigma-c)^2-\sigma(\sigma\beta+\omega),\\
&\Longrightarrow \phi_\pi(\beta(c-\sigma)^2-\sigma(\beta\sigma+\omega))<(\sigma-c)^2-\sigma(\sigma\beta+\omega),\\
&\Longrightarrow \begin{cases}\phi_\pi<\frac{(\sigma-c)^2-\sigma(\sigma\beta+\omega)}{\beta(c-\sigma)^2-\sigma(\beta\sigma+\omega)} & \text{if } \beta(c-\sigma)^2-\sigma(\beta\sigma+\omega)>0,\\ \phi_\pi>\frac{(\sigma-c)^2-\sigma(\sigma\beta+\omega)}{\beta(c-\sigma)^2-\sigma(\beta\sigma+\omega)} & \text{if } \beta(c-\sigma)^2-\sigma(\beta\sigma+\omega)<0.\end{cases}
\end{aligned} \tag{B83}$$

Both $\beta(c-\sigma)^2-\sigma(\beta\sigma+\omega)$ and $(\sigma-c)^2-\sigma(\sigma\beta+\omega)$ are upward open parabolas in c, with the same roots $\sigma\pm\sqrt{\sigma(\beta\sigma+\omega)}$. With the standard calibration we use, $\sigma\beta+\omega>\sigma$, that is, $\sigma-\sqrt{\sigma(\beta\sigma+\omega)}<0$, while $c>0$. Also, we are only interested in the region where $c<\sigma$ (otherwise, the third eigenvalue is large than unity), i.e., $c<\sigma+\sqrt{\sigma(\beta\sigma+\omega)}$. Then, $\beta(c-\sigma)^2-\sigma(\beta\sigma+\omega)<0$, and the condition for ϕ_π is:

$$\phi_\pi>\frac{(\sigma-c)^2-\sigma(\sigma\beta+\omega)}{\beta(c-\sigma)^2-\sigma(\beta\sigma+\omega)}>1. \tag{B84}$$

Determinacy. Similar to the case with the forward-looking rule, the condition for determinacy is:

$$|\bar{c}_y^{MSV}|<1 \Longrightarrow |-\frac{\phi_\pi}{\sigma}c-\frac{\phi_\pi c^2}{\sigma(\sigma-c)}+\frac{c}{\sigma-c}+1|<1, \tag{B85}$$

$$\Longrightarrow |-\frac{(\phi_\pi-1)c}{\sigma-c}+1|<1. \tag{B86}$$

The inequality holds for $\phi_\pi-1>0$ and $\sigma>c$, where the solution is E-stable. If $-\frac{(\phi_\pi-1)c}{\sigma-c}+1<0$, that is, $-\frac{\phi_\pi c-\sigma}{\sigma-c}<0$, and $\phi_\pi>\frac{\sigma}{c}$, then the condition for determinacy is $-\frac{\phi_\pi c-\sigma}{\sigma-c}>-1$ or $\phi_\pi<\frac{2\sigma}{c}+1$. □

B.3 E-Stability and the MSFE Criterion

Proof. Proposition 3.0.2. **Forward-looking policy rule**. We first derive the coefficients in the agents' forecast rules and in the actual law of motion for inflation and output conditional on agents using the M_π rule:

$$\pi_t = \bar{b}_\pi\pi_{t-1}+\bar{c}_\pi y_{t-1}+\omega g_t+u_t, \tag{B87}$$

$$y_t = \bar{b}_y\pi_{t-1}+\bar{c}_y y_{t-1}+g_t, \tag{B88}$$

where the ALM coefficients are the following:

$$\bar{b}_y = b_y(\beta_\pi)^2, \tag{B89}$$

$$\bar{c}_y = c_y, \tag{B90}$$

$$\bar{b}_\pi = b_\pi(\beta_\pi)^2, \tag{B91}$$

$$\bar{c}_\pi = c_\pi, \tag{B92}$$

with b_π, c_π, b_y, and c_y being the re-defined coefficients for the standardization from (B10)–(B12) as:

$$\begin{aligned} b_y &= \frac{\sigma_\pi}{\sigma_y}\frac{1-\phi_\pi}{\sigma}, && \text{(B93)} \\ c_y &= 1, && \text{(B94)} \\ b_\pi &= \frac{\sigma_\pi}{\sigma_\pi}\left(\beta+\omega\frac{1-\phi_\pi}{\sigma}\right), && \text{(B95)} \\ c_\pi &= \frac{\sigma_y}{\sigma_\pi}\omega. && \text{(B96)} \end{aligned}$$

Deriving M$_\pi$. As in the case of the simple model, we allow our agents to be econometricians who estimate the coefficients for their learning rule as regression coefficients. Then, for M$_\pi$ the coefficient is:

$$\beta_\pi = \frac{Cov\left(\pi_t,\pi_{t-1}\right)}{Var\left(\pi_{t-1}\right)} = \frac{Cov\left(\bar{a}_\pi+\bar{b}_\pi\pi_{t-1}+\bar{c}_\pi y_{t-1},\pi_{t-1}\right)}{Var\left(\pi_{t-1}\right)},$$

denoting $\sigma_{\pi y} \equiv Cov(y,\pi)$, $\sigma_\pi^2 \equiv Var(\pi) = 1$, $\sigma_y^2 \equiv Var(y) = 1$:

$$\beta_\pi = \bar{b}_\pi + \bar{c}_\pi\sigma_{\pi y}. \qquad \text{(B97)}$$

Deriving M$_y$. Coefficient c_π^y is then determined as a regression coefficient:

$$\begin{aligned} c_\pi^y &= \frac{Cov\left(\pi_t,y_{t-1}\right)}{Var\left(y_{t-1}\right)} = \frac{Cov\left(\bar{a}_\pi+\bar{b}_\pi\pi_{t-1}+\bar{c}_\pi y_{t-1},y_{t-1}\right)}{Var\left(y_{t-1}\right)} = \\ &= \bar{b}_\pi\sigma_{\pi y}+\bar{c}_\pi. && \text{(B98)} \end{aligned}$$

E-stability. With the coefficients of the actual law of motion defined as in (B89)–(B92). For M$_\pi$ to be E-stable under least-squares learning, the following must be satisfied

$$\frac{\partial T_\beta}{\partial \beta_\pi} < 1,$$

where T_β is as in (B97). The plot of this condition is shown in Figure B1. For M$_y$ to be E-stable under least-squares learning, the following must be satisfied

$$\frac{\partial T_{c_y}}{\partial c_\pi^y} < 1.$$

With T_{c_y} as in (B98) the condition is satisfied.

Moving to **contemporaneous inflation in the policy rule**, we solve for the coefficients of the actual law of motion as:

$$\begin{aligned} \bar{b}_y &= \beta_\pi(b_y^c+b_y^f\beta_\pi), && \text{(B99)} \\ \bar{c}_y &= 1, && \text{(B100)} \\ \bar{b}_\pi &= \beta_\pi(b_\pi^c+b_\pi^f\beta_\pi), && \text{(B101)} \\ \bar{c}_\pi &= \omega, && \text{(B102)} \end{aligned}$$

where the model coefficients are adjusted as:

$$b_y^f = \frac{\sigma_\pi}{\sigma_y}\frac{1}{\sigma}, \tag{B103}$$

$$b_y^c = -\frac{\sigma_\pi}{\sigma_y}\frac{\phi_\pi}{\sigma}, \tag{B104}$$

$$c_y = \frac{\sigma_y}{\sigma_y} = 1, \tag{B105}$$

$$b_\pi^f = \frac{\sigma_\pi}{\sigma_\pi}\left(\beta + \frac{\omega}{\sigma}\right) = \beta + \frac{\omega}{\sigma}, \tag{B106}$$

$$b_\pi^c = -\frac{\sigma_\pi}{\sigma_\pi}\frac{\omega\phi_\pi}{\sigma} = -\frac{\omega\phi_\pi}{\sigma}, \tag{B107}$$

$$c_\pi = \omega. \tag{B108}$$

The solutions for M_π and M_y look the same as in (B97) and in (B98), respectively, where the ALM coefficients are given in (B99)–(B102).

MSFE. For the agents to use M_π as the forecasting rule in equilibrium, the rule must produce a better forecast than the alternative M_y. We compare the quality of the forecasts based on the mean squared forecast error criterion. In the calculations below we denote the correlation between output and inflation as R. The correlation is then $R = \sigma_{\pi y}$, with $\sigma_{\pi y} \equiv Cov(\pi, y)$, $\sigma_\pi \equiv \sqrt{Var(\pi)} = 1$, $\sigma_y \equiv \sqrt{Var(y)} = 1$. We start with the mean forecast error of M_π. The forecast error of M_π is the difference between the forecast and actual inflation:

$$\begin{aligned} &: \quad e_t^\pi = (\beta_\pi - \bar{b}_\pi)\pi_{t-1} - \bar{c}_\pi y_{t-1} + \mu_t \\ &= \quad (\bar{b}_\pi + \bar{c}_\pi R - \bar{b}_\pi)\pi_{t-1} - \bar{c}_\pi y_{t-1} + \mu_t \\ &= \quad \bar{c}_\pi R\pi_{t-1} - \bar{c}_\pi y_{t-1} + \mu_t, \end{aligned} \tag{B109}$$

where $\mu_t \equiv \frac{\omega\sigma_g}{\sigma_\pi} g_t + \frac{\sigma_u}{\sigma_\pi} u_t$ is a composite of standardized shocks

$$\begin{aligned} &: \quad MSFE_\pi = E_t(e_t^\pi)^2 = E_t[\bar{c}_\pi R(\pi_{t-1}) - \bar{c}_\pi(y_{t-1})]^2 \\ &: \quad = E_t[\bar{c}_\pi^2 R^2(\pi_{t-1})^2 - 2\bar{c}_\pi R\bar{c}_\pi(\pi_{t-1} - \bar{\pi})(y_{t-1}) + \bar{c}_\pi^2(y_{t-1})^2 + 1] \\ &: \quad = \bar{c}_\pi^2(R^2 + 1 - 2R\sigma_{\pi y}) + 1 \\ &: \quad = \bar{c}_\pi^2(R^2 + 1 - 2R^2) + 1 \\ &: \quad = \bar{c}_\pi^2(1 - R^2) + 1, \end{aligned} \tag{B110}$$

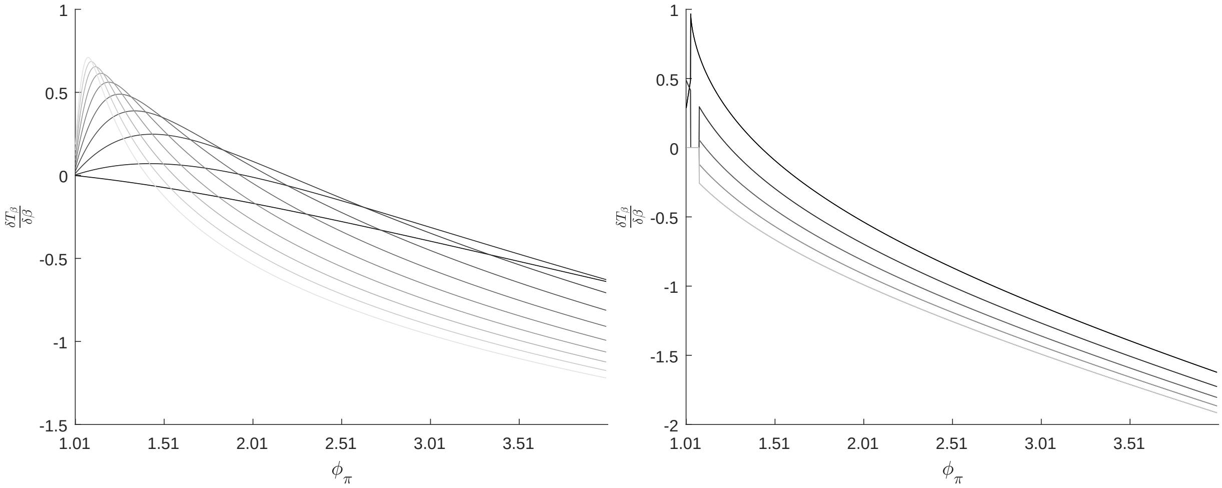
where the unity term refers to $\sigma_\mu = 1$. Similarly, the forecast error of M_y is:

$$\begin{aligned} &: \quad e_t^y = \left(c_y^y - \bar{c}_\pi\right) y_{t-1} - \bar{b}_\pi \pi_{t-1} + \mu_t = \\ &: \quad = \bar{b}_\pi \sigma_{\pi y} y_{t-1} - \bar{b}_\pi \pi_{t-1} + \mu_t = \\ &: \quad MSFE_y = E[\bar{b}_\pi[R(y_{t-1}) - (\pi_{t-1})]^2 + 1 \\ &: \quad = \bar{b}_\pi^2 \sigma_\pi^2 [1 - R^2] + 1. \end{aligned} \tag{B111}$$

We are looking for the conditions under which $MSFE_\pi < MSFE_y$:

$$\bar{c}_\pi^2(1 - R^2) < \bar{b}_\pi^2(1 - R^2), \tag{B112}$$

Figure B1: E-Stability



(a) Forward-Looking Rule ***(b) Contemporaneous Rule***

Note: The figure for the forward-looking rule is drawn for r in the range [0.1 2], where each line corresponds to a different r – the darker is the line, the larger is r. The figure for the contemporaneous rule is drawn for r in the range [0.1 1]

denoting $\bar{\Gamma} = \frac{\bar{b}_\pi}{\bar{c}_\pi}$. And the criterion is simply:

$$\bar{\Gamma}^2 > 1. \tag{B113}$$

□

Variances and Covariance

In this section, we derive the variance and covariance of the original, non-standardized variables. The adjustments necessary for standardization are mentioned in the text. The procedure is similar to Adam (2005). With the actual law of motion as in (B87) and (B88) under different policy rules we derive the variances and covariance of output and inflation. Denote $z_t = (\pi_t, y_t)'$ and $B = \begin{pmatrix} \bar{b}_\pi & \bar{c}_\pi \\ \bar{b}_y & \bar{c}_y \end{pmatrix}$, $U_t = (u_t, g_t)'$. Then the (B87) can be represented as:

$$z_t = Bz_{t-1} + U_t, \tag{B114}$$

where for the purposes of the variance and covariance calculation we have dropped the unimportant constants. Denote $r^2 \equiv \frac{\sigma_u^2}{\sigma_g^2}$ and $\Omega \equiv E(ug)^2 = \sigma_g^2 \begin{pmatrix} r^2 + \omega^2 & \omega \\ \omega & 1 \end{pmatrix}$ is the variance-covariance matrix of the shocks. We take the variance of (B114) to get:

$$\mathscr{E} = B\mathscr{E}B' + \Omega, \tag{B115}$$

where $\mathscr{E} = E(zz')$. Vectorizing matrix $\mathscr{E}$:

$$\begin{aligned} vec(\mathscr{E}) &= (B \otimes B)vec(\mathscr{E}) + vec(\Omega) \\ &= (I - B \otimes B)^{-1} vec(\Omega). \end{aligned} \tag{B116}$$

To obtain the covariance with lagged values, we multiply both sides of (B114) by z_{t-1} and take expectations to get:

$$\Gamma = B\mathscr{E}, \tag{B117}$$

where $\Gamma = E(z_t z'_{t-1})$ is the covariance matrix with the lagged values and

$$vec(\Gamma) = (I \otimes B)vec(\mathscr{E}). \tag{B118}$$

Then the variances and covariances are as follows:

$$\begin{aligned}
\sigma_{y\pi} &= vec(\mathscr{E})[2], &\text{(B119)}\\
\sigma_{\pi}^2 &= vec(\mathscr{E})[1], &\text{(B120)}\\
\sigma_{y}^2 &= vec(\mathscr{E})[4], &\text{(B121)}\\
E(\pi_t\pi_{t-1}) &= vec(\Gamma)[1], &\text{(B122)}\\
E(\pi_t y_{t-1}) &= vec(\Gamma)[3]. &\text{(B123)}
\end{aligned}$$

Appendix C: Weights

Proof. Proposition 3.0.3. Considering (31) and (32) and using the fact that $\bar{c}_\pi > 0$ and $\bar{b}_\pi \neq 0$:[10]

$$\begin{aligned}
m_\pi > m_y \quad &: \quad \frac{\kappa}{\bar{b}_\pi^2\left(1-R^2\right)}\frac{\bar{c}_\pi\sigma_y - \bar{b}_\pi R}{\bar{c}_\pi} < \frac{\kappa}{\bar{c}_\pi^2\left(1-R^2\right)}\frac{\bar{b}_\pi - \bar{c}_\pi R}{\bar{b}_\pi},\\
&: \quad \begin{cases} \frac{\bar{c}_\pi - \bar{b}_\pi R}{\bar{b}_\pi} < \frac{\bar{b}_\pi - \bar{c}_\pi R}{\bar{c}_\pi} & \text{if } \bar{b}_\pi > 0,\\ \frac{\bar{c}_\pi - \bar{b}_\pi R}{\bar{b}_\pi} > \frac{\bar{b}_\pi - \bar{c}_\pi R}{\bar{c}_\pi} & \text{if } \bar{b}_\pi < 0,\end{cases}\\
&: \quad \left(\bar{c}_\pi - \bar{b}_\pi R\right)\bar{c}_\pi < \bar{b}_\pi\left(\bar{b}_\pi - \bar{c}_\pi R\right),\\
&: \quad \bar{c}_\pi^2 - \bar{c}_\pi\bar{b}_\pi R < \bar{b}_\pi^2 - \bar{b}_\pi\bar{c}_\pi\sigma_y R,\\
&: \quad \bar{c}_\pi^2 < \bar{b}_\pi^2,\\
&: \quad \bar{\Gamma}^2 > 1.
\end{aligned}$$

□

Proof. Proposition 3.0.4. Rearranging (31)

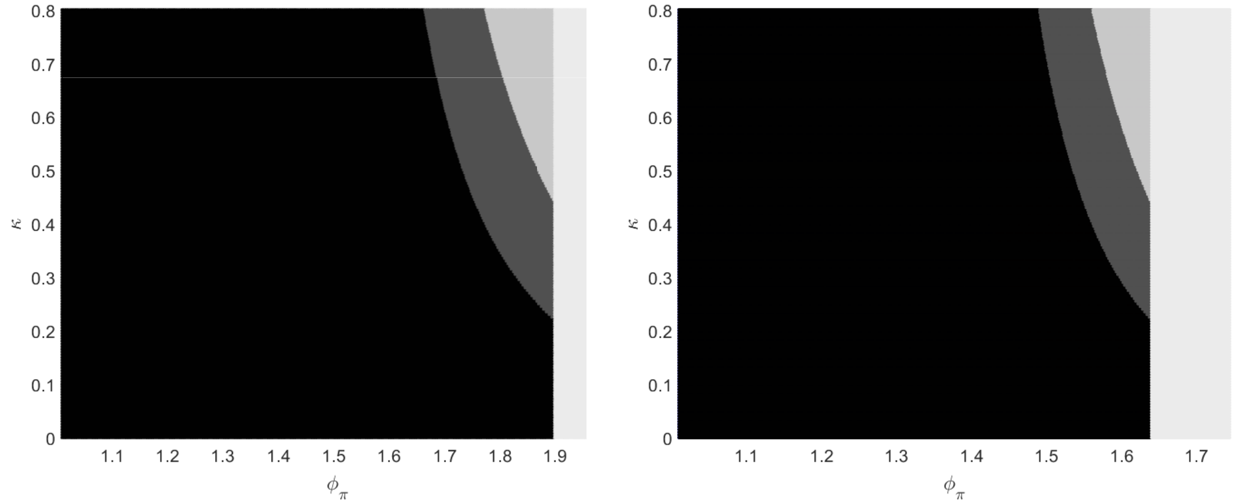
$$\begin{aligned}
&: \quad \frac{\kappa}{\bar{c}_\pi^2\left(1-R^2\right)}\frac{\bar{b}_\pi - \bar{c}_\pi R}{\bar{b}_\pi} < 1,\\
&: \quad \kappa < \frac{\bar{b}_\pi\bar{c}_\pi^2\left(1-R^2\right)}{\bar{b}_\pi - \bar{c}_\pi R} = \frac{\left(1-R^2\right)\bar{c}_\pi^2}{1 - R\frac{\bar{c}_\pi}{\bar{b}_\pi}},
\end{aligned}$$

where in the second step we again use the fact that (28) implies $|\bar{b}_\pi| > \bar{c}_\pi$, which means that even for $\bar{b}_\pi < 0$, $\frac{\bar{b}_\pi - \bar{c}_\pi R}{\bar{b}_\pi} > 0$. $R \in [-1,1]$, that is, the maximum of the expression $-\bar{c}_\pi R$, linear in R, is $\bar{c}_\pi < |\bar{b}_\pi|$. As was shown in Appendix A.3, the condition for an inner solution is sufficient for the shock to have a positive weight in the forecast. □

[10] With $\bar{c}_\pi > 0$, $\bar{b}_\pi = 0$ means violation of (28), hence with $\bar{b}_\pi = 0$ M_π cannot be the equilibrium choice.

Appendix D: Weight on Inflation

Figure D1: Model Selection under Sparse Weights, M_π.



(a) Forward-Looking Rule ***(b) Contemporaneous Rule***

CNB Working Paper Series (since 2017)		
8/2018	Volha Audzei Sergey Slobodyan	*Sparse restricted perception equilibrium*
7/2018	Tibor Hlédik Jan Vlček	*Quantifying the natural rate of interest in a small open economy – The Czech case*
6/2018	Václav Brož Michal Hlaváček	*What drives distributional dynamics of client interest rates on consumer loans in the Czech Republic? A bank-level analysis*
5/2018	Lukáš Pfeifer Martin Hodula	*Profit-to-provisioning approach for setting the countercyclical capital buffer: The Czech example*
4/2018	Ivan Sutóris	*Asset prices in a production economy with long run and idiosyncratic risk*
3/2018	Michal Franta	*The likelihood of effective lower bound events*
2/2018	Soňa Benecká Ludmila Fadejeva Martin Feldkircher	*Spillovers from euro area monetary policy: A focus on emerging Europe*
1/2018	Jan Babecký Clémence Berson Ludmila Fadejeva Ana Lamo Petra Marotzke Fernando Martins Pawel Strzelecki	*Non-base wage components as a source of wage adaptability to shocks: Evidence from European firms, 2010-2013*
16/2017	Oxana Babecká Kucharčuková Renata Pašalićová	*Firm investment, financial constraints and monetary transmission: An investigation with Czech firm-level data*
15/2017	Václav Brož Lukáš Pfeifer Dominika Kolcunová	*Are the risk weights of banks in the Czech Republic procyclical? Evidence from wavelet analysis*
14/2017	Jan Babecký Kamil Galuščák Diana Žigraiová	*Wage dynamics and financial performance: Evidence from Czech firms*
13/2017	Peter Claeys Bořek Vašíček	*Transmission of uncertainty shocks: Learning from heterogeneous responses on a panel of EU countries*
12/2017	Michal Dvořák Luboš Komárek Zlatuše Komárková Adam Kučera	*Longer-term yield decomposition: An analysis of the Czech government yield curve*
11/2017	Jan Brůha Michal Hlaváček Luboš Komárek	*House prices and household consumption: The case of the Czech Republic*
10/2017	Jan Brůha Oxana Babecká Kucharčuková	*An empirical analysis of macroeconomic resilience: The case of the great recession in the European Union*
9/2017	Simona Malovaná Dominika Kolcunová Václav Brož	*Does monetary policy influence banks' perception of risks?*
8/2017	Simona Malovaná	*Banks' capital surplus and the impact of additional capital requirements*
7/2017	František Brázdik	*A BVAR model for forecasting of Czech inflation*

	Michal Franta	
6/2017	Jan Brůha Moritz Karber Beatrice Pierluigi Ralph Setzer	*Understanding rating movements in euro area countries*
5/2017	Jan Hájek Roman Horváth	*International spillovers of (un)conventional monetary policy: The effect of the ECB and US Fed on non-Euro EU countries*
4/2017	Jan Brůha Jaromír Tonner	*An exchange rate floor as an instrument of monetary policy: An ex-post assessment of the Czech experience*
3/2017	Diana Žigraiová Petr Jakubík	*Updating the ultimate forward rate over time: A possible approach*
2/2017	Mirko Djukić Tibor Hlédik Jiří Polanský Ljubica Trajčev Jan Vlček	*A DSGE model with financial dollarization – the case of Serbia*
1/2017	Michal Andrle Miroslav Plašil	*System priors for econometric time series*

CNB RESEARCH AND POLICY NOTES (SINCE 2017)

1/2017	Mojmír Hampl Tomáš Havránek	*Should inflation measures used by central banks incorporate house prices? The Czech National Bank's approach*
2/2017	Róbert Ambriško Vilma Dingová Michal Dvořák Dana Hájková Eva Hromádková Kamila Kulhavá Radka Štiková	*Assessing the fiscal sustainability of the Czech Republic*

CNB ECONOMIC RESEARCH BULLETIN (SINCE 2017)

May 2018	*Risk-sensitive capital regulation*
November 2017	*Effects of monetary policy*
May 2017	*Trade and external relations*

Czech National Bank
Economic Research Division
Na Příkopě 28, 115 03 Praha 1
Czech Republic
phone: +420 2 244 12 321
fax: +420 2 244 12 329
http://www.cnb.cz
e-mail: research@cnb.cz
ISSN 1803-7070